# Management of Uncertainty within Estimation in Dynamical Context Application to MEMS

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***Abstract*— A probabilistic approach is proposed to manage uncertainty when dealing with estimation in dynamical models. The approach utilizes a linear integral transformation and relies on McShane's theory of stochastic differential equations. The starting point is a knowledge-based model where the estimation problem is set. The approach is quite general, it is explained here by the light of an engineering application.**

## I. Introduction

The starting point in the resolution of an estimation problem is the modeling: mathematical description of the problem. When the model comes from physics, it is said knowledge-based model, as opposed to black-box model. A model consists of a set of relations between some quantities among them appears the one to be estimated. The term "dynamical" in the title refers to the evolution in time of some quantities, and means that the model comprises at least one dynamical relation. In some model, the quantities that when fixed cause the others to be uniquely determined are called "model's data" such as the imposed conditions on the solution of an ordinary or partial differential equation, observations, controls, parameters, etc. Often some of the model's data are unknown, this is expressed by the term "uncertainty". Prior information about some unknown can be inquired. It will consist of statistics that approximate some of its moments, if it is random or of some set where it takes its values if it is deterministic. So the estimation method has to come face to face with the propagation of the information from the unknown model's data to the quantity to be estimated. The management of uncertainty within dynamical models implies stochastic processes calculus; McShane's theory is used in this instance [1].

The application here is about the robust design for a microaccelerometer, as regards to the uncertainties in the fabrication process: manufacturing tolerances and errors. Actually the effect of manufacturing tolerances and errors in a microelectromechanical system (MEMS) is more significant than in a macro-scaled one; a robust design of a MEMS passes through the study of such an effect. Here we assume that uncertainty in the fabrication process concerns only the thickness $h$ of supporting beams in the microaccelerometer. The latter is also constituted of a vibrating plate, and electrodes for driving and sensing. It is assumed that the vibrating plate oscillates only in one direction, the $x$-axis. When an acceleration is applied the beams which are about the $y$-axis flex; their deformation is denoted $d(t,y)$, $0 \leq y \leq l$, where $l$ is the beam length. It is well known that the acceleration, which is the quantity to be measured by the accelerometer, is proportional to the plate displacement $d(t,l)$. So the former is to be estimated when $h$ varies in a given interval.

The model for this estimation problem is the following nonhomogeneous linear partial differential equation with homogeneous linear imposed conditions.

$$\rho h \frac{\partial^2 d}{\partial t^2}(t,y) + \mu \frac{\partial d}{\partial t}(t,y) + \frac{\gamma h^3}{12(1-\nu^2)} \frac{\partial^4 d}{\partial y^4}(t,y) = f(t,y) \tag{1}$$

$$d(0,y) = \frac{\partial d}{\partial t}(0,y) = 0 \tag{2}$$

$$d(t,0) = \frac{\partial d}{\partial y}(t,0) = 0 \tag{3}$$

$$\frac{\partial d}{\partial y}(t,l) = \frac{\partial^3 d}{\partial y^3}(t,l) = 0 \tag{4}$$

$\rho$, $\mu$, $\gamma$, and $\nu$ are respectively the material density of the plate, the air viscosity, Young's modulus of elasticity and Poisson's ratio. $f(t,y)$ is a forcing function such that

$$f(t,y) = \lambda \ \operatorname{sign}\left(\frac{\partial d}{\partial t}(t,y)\right), \tag{5}$$

where $\lambda$ is a positive real. When we model $h$ by a random variable of a given probability density on a given interval, $d(t,l)$ becomes a random process. In this instance, estimation should concern its probability density.

The paper is organized as follows. Section 2 is the modeling of the estimation problem by a stochastic differential equation (SDE). Section 3 is the density estimation, of Monte-Carlo type using a demarginalization technique. Section 4 concludes the paper.

## II. Modeling

Consider the linear integral transformation

$$D(t) = \int_0^l K(y) d(t,y) dy. \tag{6}$$

The relevant integral (6) is assumed to exist as well as a convergent inversion formula. The function $K(y)$ is called the kernel; it is to be constructed in the following. To apply the transformation on (1), we have to calculate

$$\int_0^l K(y) O(d)(t,y) dy$$

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where $O(d)(t,y) = \frac{\partial^4 d}{\partial y^4}(t,y)$. Partial integration gives

$$\int_0^l K(y)O(d)(t,y)dy = \left[K\frac{\partial^3 d}{\partial y^3} - K'\frac{\partial^2 d}{\partial y^2} + K''\frac{\partial d}{\partial y} - K^{(3)}d\right]_{y=0}^{y=l} + \int_0^l K^{(4)}(y)d(t,y)dy$$

Regarding the imposed conditions on $d(t,y)$ in (3-4), if $K(y)$ has the following imposed conditions

$$K(0) = 0, \tag{7}$$

$$K'(0) = K'(l) = 0, \tag{8}$$

$$K^{(3)}(l) = 0, \tag{9}$$

then

$$\int_0^l K(y)O(d)(t,y)dy = \int_0^l O(K)(y)d(t,y)dy. \tag{10}$$

If $K(y)$ is chosen so that

$$O(K)(y) = \beta^4 K(y), \tag{11}$$

where $\beta$ is some real, then (1) transforms into an ordinary differential equation:

$$\rho h\ddot{D}(t) + \mu\dot{D}(t) + \frac{\beta^4\gamma h^3}{12(1-\nu^2)}D(t) = F(t), \tag{12}$$

for the unknown integral transform $D(t)$ of $d(t,y)$, where $F(t) = \int_0^l K(y)f(t,y)dy$. In the following solutions of (11) for the different possible values of $\beta$ permitting to satisfy (7-9) are sought. We find the following discrete set for $\beta$, and the corresponding solutions $K(y)$ of (11):

$$\begin{aligned} \beta_i = & \quad \frac{x_i}{l}, \\ K_i(y) = & \quad A(\cos(\beta_i y) - \cosh(\beta_i y)) + \\ & \quad Ay_i(\sin(\beta_i y) - \sinh(\beta_i y)), \end{aligned}$$

where $(x_i, y_i)$ is a point of the plane $(x,y) \in \mathscr{R}^2$ where $y = \tan(x)$ and $y = -\tanh(x)$ intercept, and $A$ is some real.

Now, the function $K(y)$ may be superimposed to construct a solution for (1-4), i.e. a solution which matches the given forcing function, boundary conditions and initial conditions. In fact, by construction this function does match the given boundary conditions (3-4); in addition, $\{K_i(y)\}_i$ form an orthogonal set, convenient for expanding $f(t,y)$ and $d(t,y)$ in the form

$$f(t,y) \approx \sum_{i\in I_1} f_i K_i(y), \quad f_i = \frac{(K_i, f)}{(K_i, K_i)}, \tag{13}$$

$$d(t,y) \approx \sum_{i\in I_2} d_i K_i(y), \quad d_i = \frac{(K_i, d)}{(K_i, K_i)}, \tag{14}$$

where $(u_1, u_2) = \int_0^l u_1(y)^* u_2(y)dy$ (* denotes the complex conjugate). (13) and (14) are approximations to $f(t,x)$ and $d(t,x)$ respectively, in terms of orthogonal functions, in the least mean-square error sense. Note that the coefficients of the expansion are independent of the number of terms in the sum.

We construct a solution for (1-4) from each trial function $K(y)$, i.e. considering just one couple $\beta, K(y)$ as follows:

$$d(t,y) = \frac{(K,d)}{(K,K)}K(y); \tag{15}$$

as $K(y)$ is real, $(K,d) = \int_0^l K(y)d(t,y)dy = D(t)$. So

$$d(t,y) = \frac{D(t)}{(K,K)}K(y), \tag{16}$$

and this constitutes the inverse formula of the integral transformation (6).

Initial conditions (2) imply that $D(0) = 0$, and $\dot{D}(0) = 0$. In addition, (16) and (5) imply that

$$f(t,y) = \lambda \ \mathrm{sign}(\dot{D}(t)K(y)),$$

and

$$F(t) = \lambda \ \mathrm{sign}(\dot{D}(t)) \int_0^l K(y) \ \mathrm{sign}(K(y))dy.$$

We recall that the quantity of interest is the plate displacement $d(t,l)$

$$d(t,l) = \frac{D(t)}{(K,K)}K(l),$$

when the beam thickness $h$ varies in a given interval.

If we set $X^1(t) = D(t)$ and $X^2(t) = \dot{D}(t)$, we obtain the following SDE as a model for our estimation problem:

$$\begin{aligned} &\dot{X}^1 = X^2, \\ &\dot{X}^2 = -\frac{\beta^4\gamma h^2}{12\rho(1-\nu^2)}X^1 \\ &-\frac{\mu}{\rho h}X^2 + \frac{\lambda\int_0^l K(y) \ \mathrm{sign}(K(y))dy}{\rho h} \ \mathrm{sign}(X^2), \\ &X^1(0) = 0, \quad X^2(0) = 0, \end{aligned} \tag{17}$$

where $h$ is a random variable of a given probability density.

## III. Estimation

The objective of this section is to estimate the probability density functions $p(t,\varepsilon)$ and $p(t,\upsilon)$ relative respectively to the stochastic processes $X^1$ and $X^2$ in (17), at some time $t$. Consider the Euler discretization of the SDE (17) at instants $t_n$. It is worth noting that discretization instants are not necessarily equally spaced.

$$X^1_{n+1} - X^1_n = (t_{n+1} - t_n)X^2_n, \tag{18}$$

$$X^2_{n+1} - X^2_n = -\frac{\beta^4\gamma h^2}{12\rho(1-\nu^2)}(t_{n+1} - t_n)X^1_n - \frac{\mu}{\rho h}(t_{n+1} - t_n)X^2_n + \frac{\lambda\int_0^l |K(y)|dy}{\rho h}(t_{n+1} - t_n) \ \mathrm{sign}(X^2_n), \tag{19}$$

(18) implies that

$$p_{X^1_{n+1}|X^1_n, X^2_n} = \delta\left(X^1_{n+1} - \left(X^1_n + (t_{n+1} - t_n)X^2_n\right)\right).$$

(19) implies that

$$p_{X^2_{n+1}|X^1_n,X^2_n} = \left. \frac{1}{\left|2c_1 h - \frac{c_2}{h^2}\right|} p_h \right]_{h=h(X^2_{n+1})}, \qquad (20)$$

where $p_h$ is the probability density of $h$,

$$c_1 = -\frac{\beta^4 \gamma}{12\rho(1-\nu^2)}(t_{n+1}-t_n)X^1_n,$$

and

$$c_2 = -\frac{\mu}{\rho}(t_{n+1}-t_n)X^2_n + \frac{\lambda \int_0^l |K(y)|dy}{\rho}(t_{n+1}-t_n)\ \mathrm{sign}(X^2_n).$$

(20) is the formula of the change of variables, as when conditioned on $X^1_n$ and $X^2_n$, the random variables $X^2_{n+1}$ and $h$ are related by one-to-one transformation

$$X^2_{n+1} = c_1 h^2 + \frac{c_2}{h} + c_3,$$

where $c_1$, $c_2$ are given above and $c_3 = X^2_n$. We assume a uniform distribution for $h$ on the interval $[h_{\inf}, h_{\sup}]$ (it represents manufacturing tolerances and errors):

$$p_h(h) = \frac{1}{h_{\sup} - h_{\inf}}$$

On the other hand, we have

$$\begin{aligned}
&p_{X^1_{n+1},X^2_{n+1}}(\varepsilon,\upsilon) = E\left(p_{X^1_{n+1},X^2_{n+1}|X^1_n,X^2_n}(\varepsilon,\upsilon)\right)\\
&= E\left(p_{X^1_{n+1}|X^1_n,X^2_n}(\varepsilon)p_{X^2_{n+1}|X^1_n,X^2_n}(\upsilon)\right)\\
&= E\left(\delta\left(\varepsilon - \left(X^1_n + (t_{n+1}-t_n)X^2_n\right)\right)\frac{1}{\left|2c_1 h - \frac{c_2}{h^2}\right|}p_h\right)_{h=h(\upsilon)}
\end{aligned} \qquad (21)$$

where $E$ is the mathematical expectation. Note that $c_1$ and $c_2$ are functions of the random variables $X^1_n$ and $X^2_n$. So $p_{X^1_{n+1},X^2_{n+1}}$ may be approximated by empirical mean of the expression in (21), with respect to the $X^1_n$ and $X^2_n$. Samples of theses random variables are obtained from (18-19) and from samples of $h$. The densities $p_{X^1_{n+1}}$ and $p_{X^2_{n+1}}$ are then derived by marginalization.

## IV. Illustration

In order to illustrate the material of the previous sections, the following values are assumed: $\rho = 2320$, $\gamma = 170e9$, $\nu = 0.25$, $\mu = 1000$, $\lambda = -1e6$, $h_{\inf} = 0.8e-6$, $h_{\sup} = 1.2e-6$, $l = 100e-6$ (SI Units). The plate displacement and velocity for a realization of $h$ are reported in the figure 1 and 2 respectively, corresponding to ten natural periods of the system (a period amounts around $2e-6$ second, it is denoted $T_0$). Figures 3 and 4 show approximation of $p_{X^1}(t,\varepsilon)$ and $p_{X^2}(t,\upsilon)$, at some fixed instant $t$, obtained by classical Monte Carlo technique. 3000 simulations of (18-19) during $[0,t]$ are needed for such approximation. This is our unique reference to evaluate precision of our estimation; it is also to be compared with the latter in terms of simulation cost (time and memory).

According to the notation of section 3, let's take $t_{n+1} = t$ and simulate (18-19) at instants $t_0 < t_1 < \ldots < t_n < t$. For $t - t_n = T_0/10$ and just 10 simulations, we obtain the approximations of $p_{X^1(t)}$ and $p_{X^2(t)}$ shown in figures 5 and 6 respectively. For 60 simulations we obtain the result shown in figures 7 and 8. As suggested, the marginalization of formula (21) on $\upsilon$ accompanied with empirical mean formula, give an approximation of $p_{X^1(t)}$. Up to the inverse of the number of simulations, it is a Dirac series marked on the figures 5 and 7 by symbol $+$. By the same reasoning the approximation of $p_{X^2(t)}$ shown in figures 6 and 8, at some point of its support is, up to a normalization constant, the sum of point ordinates relative to every small curve whose support contains that point (these small curves appear clearly if we zoom in the plot). Even with such a small number of simulations of (18-19), and thus highly reduced time and memory consumption, the result conforms with the reference and proves to my satisfaction the performance of the approach.

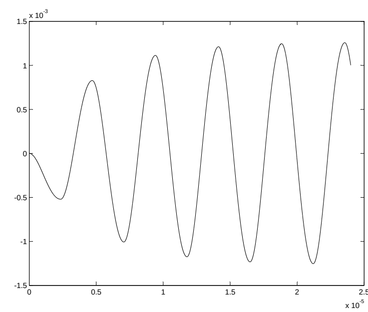


Fig. 1. Plate Displacement

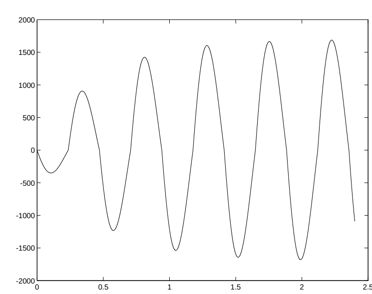


Fig. 2. Plate Velocity

## V. Conclusion

A probabilistic approach is proposed to manage uncertainty for estimation in dynamical models, when illustrated on an engineering application. The crucial task within our approach is the modeling: transformation of the knowledge-based model, where the estimation problem is set, to a stochastic differential equation. Then, approximating the

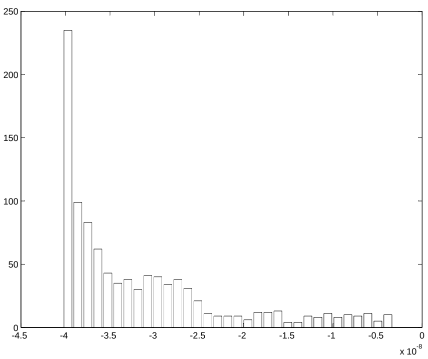


Fig. 3. Monte Carlo Approximation of $p_{X^1(t)}$

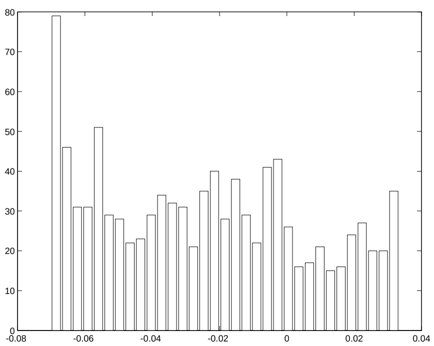


Fig. 4. Monte Carlo Approximation of $p_{X^2(t)}$

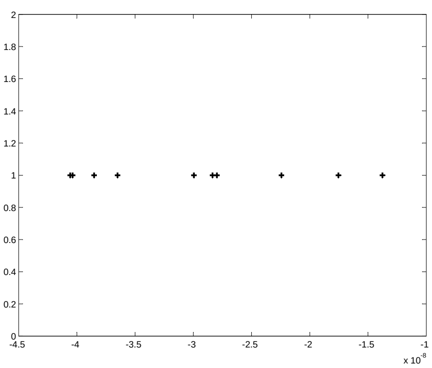


Fig. 5. Approximation of $p_{X^1(t)}$

probability density of its solution achieves the estimation. For modeling, an original operational technique is used; the latter does not apply universally, but is often likely to work. Density approximation is of Monte-Carlo type and uses a demarginalization formula. Finally, it is worth noting that the obtained SDE is linear, but this does not affect the generality of our approach.

## REFERENCES

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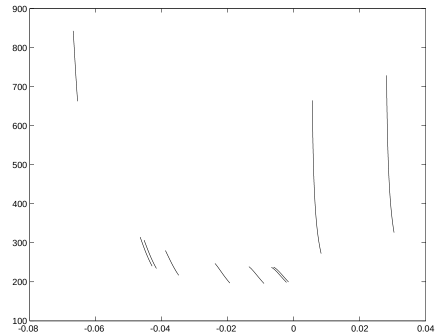


Fig. 6. Approximation of $p_{X^2(t)}$

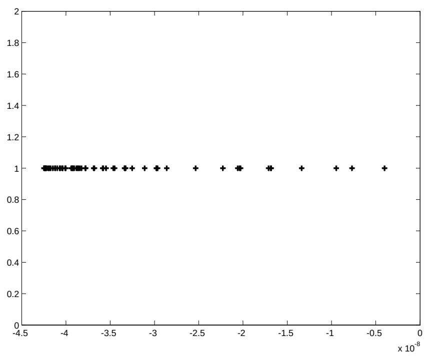


Fig. 7. Approximation of $p_{X^1(t)}$

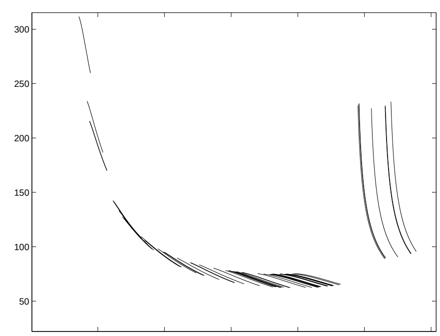


Fig. 8. Approximation of $p_{X^2(t)}$

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