Methodologies for Analyzing Equilibria in Wireless Games

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Abstract

Under certain assumptions in terms of information and models, equilibria correspond to possible stable outcomes in conflicting or cooperative scenarios where rational entities interact. For wireless engineers, it is of paramount importance to be able to predict and even ensure such states at which the network will effectively operate. In this article, we provide non-exhaustive methodologies for characterizing equilibria in wireless games in terms of existence, uniqueness, selection, and efficiency.

I. Introduction

The major works by Von Neumann, Morgenstern and Nash are recognized as real catalyzers for the theory of games which originates from the works by Waldegrave (1713), Cournot (1838), Darwin (1871), Edgeworth (1881), Zermelo (1913), Borel (1921) and Ville (1938). Whereas the strong developments of game theory and information theory occured approximatively at the same time of history, namely in the middle of the 20th century with the major works by Von Neumann, Morgenstern, Nash, and Shannon, it is only recently that deepened analyses have been conducted, at a significant scale, to apply game theory to communications problems. During the fifty years following the seminal work by Shannon there were only a small number of papers adopting a game-theoretic view of communication problems. To cite a few of them we have [1] where point-to-point communications are seen as a game between the channel encoder (choosing the best input distribution in terms of mutual information -MI-) and channel (choosing the worse transition probability in terms of MI), [2] where source coding is seen as a game between the source encoder and switcher (modifying the source distribution) having antagonistic objectives in terms of distorsion, [3] where the author exploits game theory for the joint signal-and-detector design using game-theoretic techniques to perform multi-parameter optimization, or also [4] where a legal encoder-decoder pair fights against a jammer. By contrast, many papers exploiting game theory for communications and especially wireless communications have been released over the past fifteen years ([5], [6], [7], [8], [9], etc.) and the phenomenon seems to gain more and more momentum. There are many reasons for this craze for game theory in the wireless community, here we will give a few technical reasons for this. Note that, in this article, we will focus on technical problems arising at the physical and medium access layers of a wireless network and not on economic aspects related to it, like the auction problem for spectrum, even though it is also an important scenario where game theory is used.

An important reason for this surge of interest in game theory in the area of wireless communications is the determinant role of certain major wireless actors like spectrum regulators. Here is an example illustrating this role: by allowing anybody to use some portions of the radio spectrum, which are referred to as unlicensed bands, spectrum regulators have naturally created a game. Wireless devices (or groups of devices) using these bands interact with each other by exploiting the available spectral resources. An other scenario of paramount importance where game theory is naturally the dominant paradigm to analyze interactions between devices consists of networks of terminals equipped with a cognitive radio. Indeed, as spectrum congestion has become a more and more critical issue, the Federal Communications Commission has released an important report [10] providing a legal framework for deploying such networks. Another factor for the great interest of the wireless community in game theory is that considering a terminal as an intelligent entity capable of observing and reacting sufficiently rapidly has become a more and more realistic assumption with the significant progresses of signal processing (e.g., spectrum sensing algorithms and dramatic increase of admissible computational complexity). Also, for a long time, wireless networks were based on the single-user channel paradigm introduced by Shannon but networks have evolved and multi-user channel models are now considered as a key element for increasing reliability, rates and security of wireless communications. In such networks, terminals share common resources like energy, power, routes, space, spectrum, time, etc, which implies potential interaction between them and game theory is, by definition, the branch of mathematics studying interaction between intelligent entities. Roughly speaking, the born of multiuser networks appears to be a milestone delineating two eras: the era were game theory was used to *analyze* or interpret communication systems and the era where it is used to *design* or construct communication system.

When inspecting the works applying game theory to wireless networks, there is generally a quite significant part of them dedicated to analyzing the issues related to the notion of equilibrium. This is not surprising since this concept is instrumental for wireless engineers. For instance, the existence of an equilibrium can allow them to predict the effective operating state(s) of the network. Most often, the type of equilibrium studied is the Nash equilibrium (NE) (see e.g., the recent papers in [11]), which corresponds to the minimum condition of stability for an equilibrium (i.e., it is robust to a single deviation). Whereas there are many papers where issues, such as existence and uniqueness of an equilibrium in a given wireless game, are well treated, there is no paper attempting to provide a complete methodology in order to analyze equilibria in wireless games. It is therefore difficult to have a complete picture of the existing theorems and techniques available to analyze these equilibria. One of the purposes of this article is precisely to contribute to building this picture by giving methodologies for studying equilibria in non-cooperative wireless games. One of the main goals of this article is to answer typical questions such as: Does the network have a pure NE (Sec. III)? If yes, is it unique (Sec. IV)? If it does not exist, what can be done? On the other hand, if there are several equilibria, how to select one of them (Sec. V)? If the selected equilibrium is found to be inefficient in a given sense of efficiency measure, how can the game be modified to improve its efficiency (Sec. VI)? This article will be structured (Sec. III–VI) according to this list of questions. These sections will be preceded by a section introducing the notion of equilibrium (Sec. II) and followed by a section showing to what extent the provided methodologies can be applied to other types of equilibrium (Sec. VII). Throughout the article, more emphasis will be made on the methodology rather than attempting to enumerate all the main results available. For each main issue addressed (e.g., existence) some useful definitions, theorems and techniques will be provided and concisely illustrated by a typical wireless game at the physical or medium access layer.

II. The concept of equilibrium

In this article, we will mainly focus on a certain type of games namely static strategic (form) non-cooperative games with complete information (SNG) and finite number of players. Strategic or normal form games consist of a triplet $\mathcal{G} = (\mathcal{K}, \{\mathcal{S}_i\}_{i\in\mathcal{K}}, \{u_i\}_{i\in\mathcal{K}})$ where $\mathcal{K} = \{1, ..., K\}$ is a finite set of players, $\forall i \in \mathcal{K}$, $\mathcal{S}_i$ is the set of strategies of player i and u_i his utility (payoff) function; concerning the notations we will often use $\underline{s} = (s_1, ..., s_K)$ to refer to a vector and $\mathcal{S} = \prod_{i=1}^{K} \mathcal{S}_i$ to refer to the Cartesian product of sets. The games are non-cooperative in the sense that each player/user wants to selfishly maximize his own utility u_i over his strategy set $\mathcal{S}_i$. The assumption of complete information means that every player knows the triplet $\mathcal{G}$. Further, they are implicitly assumed to be rational in the sense of Savage [12] i.e., each player does what is best for him and rationality of the players is assumed to be common knowledge. Even though we restrict our attention to SNGs, the methodology provided here can be, to a large extent, applied to other types of games, which will be discussed in Sec. VII. Also, most of our attention will be dedicated to pure-strategy Nash equilibria (pure NE), which correspond to the minimum condition of stability that is, the NE strategy profiles are stable to a single deviation. Indeed, a pure NE is defined as follows:

Definition 1 (NE): The vector $\underline{s}^*$ *is a (pure) NE for* $\mathcal{G}$ *if* $\forall i \in \mathcal{K}, \forall s_i' \in \mathcal{S}_i,\ u_i(s_i^*, \underline{s}_{-i}^*) \geq u_i(s_i', \underline{s}_{-i}^*)$ *where we use the notation* $\underline{s}_{-i}$ *to refer to the strategy profile of all players except for player* i*.*

An NE is therefore stable to a single deviation. In some contexts, for example, in the context of population (large number of players), a stronger condition of stability can be required. This is the case of evolutionary stable equilibria, which are stable to the deviation of a fraction of a population [13]. In the core of this article only pure, mixed and correlated equilibria will be mentioned. Indeed, one has to distinguish between pure, mixed, and correlated strategies. As already seen, for a player, choosing a pure strategy consists in picking one element in his set of possible actions. Implementing a mixed strategy consists in associating a probability with each of the possible actions and run the series of actions generated by the corresponding lottery. For example, a transmitter can decide to transmit at full power or not at all, by following the realizations of a Bernouilli random variable. A mixed Nash equilibrium is therefore a vector of probability distributions verifying the Nash property in Def. 1 where s_i represents a distribution, denoted by $\underline{q}_i$, and u_i has to be replaced with the expected utility for player i: $\tilde{u}_i(\underline{q}_1, ..., \underline{q}_K) = \sum_{\underline{s} \in \mathcal{S}} p(\underline{s}) u_i(\underline{s})$ where $p(\underline{s}) = \prod_{j \in \mathcal{K}} q_j(s_j)$ and $q_j(s_j)$ is the probability with which player j chooses the action $s_j \in \mathcal{S}_j$. In contrast with mixed equilibria, in correlated equilibria [14] the lotteries used by the players can be correlated (by coordination signals) that is, in general, we do not have $q(\underline{s}) = \prod_i q_i(s_i)$.

III. Existence

Mathematically speaking, proving the existence of an equilibrium amounts to proving the existence of a solution to a fixed-point problem [15]; this is why so much effort is made to derive fixed-point theorems (see e.g., [16], [17]) by game theorists. Therefore, it will not be surprising to see that equilibrium existence theorems are based on topological properties of the strategy sets of the players and topological and geometrical properties of their utility. This explains why the respective works by Lefschetz [18], Hopf [19], Brouwer [20] and Kakutani [21] on fixed-point theorems (**FPT**) have had an important impact in game theory. To prove the existence of an equilibrium at a problem at hand, one may always attempt to derive an FPT, which is the most general method. However, there are many scenarios assuming usual channel models and performance metrics, and for which existing theorems are sufficient. For example, Shannon transmission rates or rate regions have desirable convexity properties that are needed in standard known theorems for equilibrium existence. The purpose of this section is to provide certain of these useful existence theorems and mention some examples where they have been applied in the literature of wireless communications.

A very useful existence theorem is a theorem stated in [22], resulting from the contributions of [23], [24], and [25]. The corresponding theorem, which will call the Debreu-Fan-Glicksberg theorem as in [22], can be applied if the users' utilities are quasi-concave and some other properties are verified. Before stating the corresponding theorem, let us review the definition of a quasi-concave function: a function ψ is quasi-concave on a convex set $\mathcal{S}$ if, for all $\alpha \in \mathbb{R}$, the upper contour set $\mathcal{U}_\alpha = \{x \in \mathcal{S}, f(x) \geq \alpha\}$ is convex. We will use the acronym **QG** to refer to games in which utilities are quasi-concave.

Theorem 2 (Debreu-Fan-Glicksberg 1952): Let $\mathcal{G}$ be an SNG. If $\forall i \in \mathcal{K}$: $\mathcal{S}_i$ is a compact and convex set; $u_i(\underline{s})$ is a continuous function in the profile of strategies $\underline{s}$ and quasi-concave in s_i; then the game $\mathcal{G}$ has at least one pure NE.

A special case of this theorem is when the utility functions are concave. In this respect, Theorem 1 by Rosen [26] for *concave* $K-$person games can be seen as a corollary of Theorem 2. Furthermore, if the utility functions are assumed to be affine functions, Nash existence theorem [15] of a mixed equilibrium in *finite* games (**FG**; $|\mathcal{K}| < +\infty$, $|\mathcal{S}_i| < +\infty$) can also be seen as a special case of Theorem 2. This is due to the fact that the mixed strategy of player $i \in \mathcal{K}$, i.e., the distribution q_i used by player i, belongs to a compact and convex set $[0, 1]$ and his (averaged) utility is an affine function of q_i.

Example 3: For the distributed energy-efficient power control (PC) problem introduced in [8], $\mathcal{G}$ is defined as follows. The strategy of user $i \in \mathcal{K}$ is his instantaneous transmit power $p_i \in [0, P_i^{\max}]$ and his utility is $u_i(\underline{p}) = \frac{f(\mathrm{SINR}_i)}{p_i}$ where $f : \mathbb{R}^+ \to [0, 1]$ is a sigmoidal efficiency function (e.g., the packet success rate) and SINR_i is the signal-to-interference plus noise ratio for user i. This game can be shown to be quasi-concave and thus has at least one pure NE.

Of course, there is no reason why utilities used in wireless games should always be quasi-concave. For instance, in the case of the energy-efficient PC game we have mentioned, the quasi-concavity property is lost when the utility is modified into $\tilde{u}_i(\underline{p}) = u_i(\underline{p}) + \alpha p_i$, which corresponds to implementing a linear pricing technique [27]. If a game can be shown to be non quasi-concave, it still can have some nice properties that ensure the existence of a pure NE. This is precisely the case if one has to deal with an S-modular game (**SMG**) [28], [29] or a potential game (**PG**) [30]. S-modular games include submodular games [31] and supermodular games [28], [29]. In what follows we give the definition, an existence theorem and an example of game for these types of games.

Definition 4 (S-modular games): The strategic game $\mathcal{G}$ is said to be supermodular (resp. submodular) if: $\forall i \in \mathcal{K}$, $\mathcal{S}_i$ is a compact subset of $\mathbb{R}$; u_i is upper semi-continuous in $\underline{s}$; $\forall i \in \mathcal{K}, \forall \underline{s}_{-i} \geq \underline{s}'_{-i}$ the quantity $u_i(\underline{s}) - u_i(s_i, \underline{s}'_{-i})$ is non-decreasing (resp. non-increasing) in s_i.

Furthermore if the utilities of the game are twice differentiable there is a simple characterization of SMGs.

Definition 5 (Characterization of S-modular games): If $\forall i \in \mathcal{K}$, u_i *is twice differentiable, then the game* $\mathcal{G}$ *is submodular (resp. supermodular) if and only if*

$$\forall (i,j) \in \mathcal{K}^2, i \neq j, \frac{\partial^2 u_i}{\partial s_i \partial s_j} \leq 0 \text{ (resp. } \geq 0). \tag{1}$$

An interpretation of a super-modular (resp. sub-modular) game is that if the other players $-i$ increase (resp. decrease) their strategy, player i has interest in increasing (resp. decreasing) his. One of the nice properties of SMGs is that they do not require convexity, concavity assumptions on the utilities to ensure the existence of an NE. The following theorem for SMGs can be found in [31] and follows from Tarski fixed-point theorem [32].

Theorem 6 (Topkis 1979): If $\mathcal{G}$ *is an SMG, then it has at least one pure NE.*

An example of supermodular games can be found in [27] where the authors uses a linear pricing technique to improve the energy-efficiency of the NE of the distributed PC game. The corresponding utilities are $\forall i \in \mathcal{K}, \tilde{u}_i(\underline{p}) = u_i(\underline{p}) + \alpha p_i$ which are not quasi-concave, as already mentioned. An example of submodular games can be found in [33] and [34].

Now we turn our attention to an other important type of games: potential games. This type of games has been introduced by Monderer and Shapley in [30].

Definition 7 (Potential games): The game $\mathcal{G}$ *is an exact (resp. ordinal) potential game if there exists a function* ϕ *such that* $\forall i \in \mathcal{K}, \forall \underline{s} = (s_i, \underline{s}_{-i}) \in \mathcal{S}, \forall s'_i \in \mathcal{S}_i$, $u_i(\underline{s}) - u_i(s'_i, \underline{s}_{-i}) = \phi(\underline{s}) - \phi(s'_i, \underline{s}_{-i})$ *(resp.* $u_i(\underline{s}) - u_i(s'_i, \underline{s}_{-i}) > 0 \ \Leftrightarrow \ \phi(\underline{s}) - \phi(s'_i, \underline{s}_{-i}) > 0$*).*

It is important to note that the potential function is independent of the user index: for every player, ϕ allows one to quantify the impact of a unilateral deviation on all the users' utilities in exact PGs while it gives the sign of the difference of utilities in ordinal PGs. Like SMGs we have the following existence theorem.

Theorem 8 (Monderer-Shapley-1 1996): If $\mathcal{G}$ *is a PG with a finite number of players, compact strategy sets, and continuous utilities, then it has at least one pure NE.*

Example 9: In [35] the authors formulate a constrained PC problem as a game where each user wants to: minimize his transmit power $p_i \in [0, P_i^{\max}]$ *subject to the constraint* $f_i(\mathrm{SINR}_i) \geq \gamma_i$*. Considering, without loss of generality* $u_i(\underline{p}) = \log(p_i)$ *for the users's cost functions, it can be checked that* $\phi(\underline{p}) = \sum_{i=1}^{K} \log(p_i)$ *is a potential function for this game.*

We see that Theorem 8 is very useful if a potential function can be found. If such a function cannot be easily found, other results can be used to check if the game is potential. As mentioned in [30], a very simple case where it is to easy to verify if the game is potential is the case where the strategy sets are intervals of $\mathbb{R}$. In this case we have the following theorem.

Theorem 10 (Monderer-Shapley-2 1996): Let $\mathcal{G}$ be a game in which the strategy sets are intervals of real numbers. Assume the utilities are twice continuously differentiable. Then $\mathcal{G}$ is a potential game if and only if

$$\forall (i,j) \in \mathcal{K}^2, \frac{\partial^2 (u_i - u_j)}{\partial s_i \partial s_j} = 0. \tag{2}$$

It is interesting to notice that in the (non-exhaustive) list of theorems stated so far, none of them requires to explicit the best responses of the players: the best response **(BR)** of player $i \in \mathcal{K}$ corresponds, by definition, to the set of strategies $\mathrm{BR}_i(\underline{s}_{-i})$ maximizing the utility of user i when the rest of the world plays $\underline{s}_{-i}$: $\mathrm{BR}_i(\underline{s}_{-i}) = \arg\max_{s_i} u_i(s_i, \underline{s}_{-i})$. In general BR_i can be a correspondence but it is a function in many wireless games addressed in the current literature of communications. If the BRs can be explicated, the existence proof boils down to proving that the BRs have a non-empty intersection, which can be very simple in some scenarios. For instance in the power allocation (PA) game of [36] where the authors study interference relay channels, the best response of the users are piecewise affine functions in the case of their amplify-and-forward protocol. The existence issue and even uniqueness issue are quite simple to analyze in such a case. To conclude this section, we summarize the methodology presented in Fig. 1. In the box "Not in this article" one could for instance, put the recent work on distributed power control [37] where the authors show that instead of proving an FPT (mathematically), the powerful model checking concept can be used to verify (with a computer but rigorously) the existence of a certain network properties or a winning strategy.

IV. UNIQUENESS

Once one is ensured that an equilibrium exists, a natural question is to know whether it is unique. This is important not only for predicting the state of the network but also crucial for convergence issues. Unfortunately, there are not so many general results on the equilibrium uniqueness. In this section, for sake of clarity, we will distinguish between two types of situations, depending whether the BR of every player can be explicated or not.

A. The BRs do not need to be explicated

A natural question would be to ask whether the Debreu-Fan-Glicksberg theorem has a counterpart for uniqueness, that is, there exists a general uniqueness theorem for quasi-concave $K-$player games. To the best of the author's knowledge, the answer is no. However, there is a powerful tool for proving the uniqueness of a pure NE when the players' utilities are concave: this tool is the uniqueness theorem derived by Rosen [26].

This theorem states that if a certain condition, called diagonally strict concavity (DSC), is met, then uniqueness is guaranteed. This theorem is as follows.

Theorem 11 (Rosen 1965): Assume that $\forall i \in \mathcal{K}$: $\mathcal{S}_i$ *is a compact and convex set;* $u_i(\underline{s})$ *is a continuous function in* $\underline{s} \in \mathcal{S}$ *and concave in* s_i. *Let* $\underline{r} = (r_1, ..., r_K)$ *be an arbitrary vector of fixed positive parameters. Define the pseudogradient of the function* $w_{\underline{r}} = \underline{r} \times \underline{u}^T$ *by* $\underline{\gamma}_{w_{\underline{r}}}(\underline{s}) = \left[r_1 \frac{\partial u_1}{\partial s_1}(\underline{s}), ..., r_K \frac{\partial u_K}{\partial s_K}(\underline{s}) \right]^T$. *If the following condition (DSC) holds*

$$\forall (\underline{s}, \underline{s}') \in \mathcal{S}^2, \underline{s} \neq \underline{s}' : \ (\underline{s} - \underline{s}') \left(\underline{\gamma}_{w_{\underline{r}}}(\underline{s}) - \underline{\gamma}_{w_{\underline{r}}}(\underline{s}') \right)^T > 0 \tag{3}$$

then the game $\mathcal{G}$ *has a unique NE.*

To illustrate this theorem we provide an example of wireless game where it has been applied successfully.

Example 12: In [38] the authors generalized the water-filling game of [39] to fast fading multiple access channels (MACs with one base station -BS- and K *mobile stations -MS-) with multiple antennas both at the transmitters and receiver (uplink case). In this game, each user wants to maximize his ergodic transmission rate by choosing the best precoding strategy. While showing that this game is concave is quite easy, proving its uniqueness is less trivial. It turns out that the DSC of Rosen has a matrix counterpart here and can be proved to be true. Proving the DSC boils down to proving that:* $\mathrm{Tr}\,(\mathbf{MN} + \mathbf{PQ}) > 0$ *where* $\mathbf{M} = \mathbf{A} - \mathbf{B}$, $\mathbf{N} = \mathbf{B}^{-1} - \mathbf{A}^{-1}$, $\mathbf{P} = \mathbf{C} - \mathbf{D}$, $\mathbf{Q} = (\mathbf{B} + \mathbf{D})^{-1} - (\mathbf{A} + \mathbf{C})^{-1}$, *and* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ *are positive matrices.*

B. When the BRs can be explicated

If the BR of every player can be expressed, it is possible to analyze their properties and for some classes of functions (or correspondences) to characterize the number of intersection points between them. A nice class of BRs is the class of standard BRs. Standard functions have been introduced by Yates in [40]; they are defined as follows.

Definition 13 (Standard functions): A vector function $g : \mathbb{R}_+^K \to \mathbb{R}_+^K$ *is said to be standard if it has the two following properties:*

- *Monotonicity:* $\forall (\underline{x}, \underline{x}') \in \mathbb{R}_+^{2K}$, $\underline{x} \leq \underline{x}' \Rightarrow g(\underline{x}) \leq g(\underline{x}')$;
- *Scalability:* $\forall \alpha > 1$, $\forall \underline{x} \in \mathbb{R}_+^K$, $g(\alpha \underline{x}) < \alpha g(\underline{x})$.

In [40] it is shown that if a function is standard then it has a unique fixed-point. Applying this result to the best responses of a game we have the following result.

Theorem 14 (Yates 1995): Assume that the best responses of the strategic-form game $\mathcal{G}$ *are standard. Then, the game has a unique NE.*

A simple wireless game where the BRs have these properties is the energy-efficient PC game introduced by [8], which we have already mentioned. The BRs can be shown to be (see e.g., [27], [41]): $\forall \in \{1, ..., K\},\ \mathrm{BR}_i(\underline{p}) = \frac{\beta^*}{|h_i|^2}\left(\sigma^2 + \sum_{j=0}^{i-1} p_j |h_j|^2\right)$, where β^* is a constant, h_i the instantaneous channel gain of user i and σ^2 the reception noise variance. It can be checked that the BRs are monotonic and scalable.

As mentioned in the end of Sec. III, if the BRs are available and quite simple to exploit, it might be possible to find their *intersection* points. The number of intersection points corresponds to the number of equilibria. A well-known game where this kind of approaches is very simple is the Cournot duopoly [42] for which the BRs of the two players are affine and intersect in a single point. A counterpart of the Cournot duopoly in wireless networks is treated, for instance, in [43].

V. Equilibrium selection

In fact, there are important scenarios where the NE is not unique. This typically happens in routing games [44], [45] and coordination games [46]. Another important scenario, where such a problem arises, corresponds to games where the choice of actions from different players is not independent, for instance non-cooperative games with correlated constraints and generalized NE [47] is one of the solution concepts. A central feature in these constrained games is that they often possess a large number of equilibria. Natural questions that arise concern the selection of an appropriate equilibrium. What can be done when one has to deal with a game having multiple equilibria? Are they some dominant equilibria? Are there some equilibria fairer and more stable than others? Obviously, the selection rule is strongly related to the used fairness criteria. As equilibrium selection is a theory in itself [48], the authors will give here a very incomplete view of the general problem and only partial answers to these questions. In fact, we will mention only three issues related to the equilibrium selection problem: how to select an equilibrium in concave games; the role of the game dynamics in the selection; the role of efficiency in the selection. These issues have been chosen here because they are strongly connected to the content of the other sections of this article.

In the case of concave games, here again, Rosen [26] gives a very neat way to tackle the problem. In [26], the author introduced the notion of *normalized equilibria*, which gives a way of selecting an equilibrium.

Definition 15 (Normalized equilibrium): Let $\mathcal{G}$ be a concave SNG, $\underline{r}$ a vector of positive parameters, $\mathcal{C} = \{\underline{s} \in \mathcal{S},\ h(\underline{s}) \geq 0\}$ is a constraint set. An equilibrium $\underline{s}^*$ of this game is said to be a normalized NE associated with $\underline{r}$ if there exists a constant λ such that $\lambda_i = \frac{\lambda}{r_i}$ where λ_i are the multipliers corresponding to the Kuhn-Tucker conditions $\lambda_i h(\underline{s}) = 0$.

This concept has been recently applied by [49] to study decentralized MACs with constraints. The impact of these constraints is to correlate the players' actions. As mentioned in the beginning of this section, there can be multiple NE in this type of games. This is precisely what happens in decentralized MACs. One of the problems that arises in such contexts is to know how a player values the fact that constraints of another player are satisfied or violated. Some extreme cases are: (i) A player is indifferent to satisfaction of constraints of other players; (ii) Common constraints: if a constraint is violated for one player then it is violated for all players. The concept of normalized equilibrium, applicable to concave games, is precisely a possible way of predicting the outcome of such a game and/or selecting one of the possible equilibria. Specifically, the authors of [49] have shown that, in their context of multiple access game with multiuser detection, the normalized equilibrium achieves maxmin fairness and is also proportionally fair (for these notions see e.g., [50]).

So far, we have always assumed static games with complete information namely the game is played in one shot, based on the fact that every player knows everything about the game. This is in this precise framework that existence and uniqueness of an NE has been discussed. Interestingly, the Nash equilibria predicted in such framework can be observed in others that are less restrictive in terms of information assumptions. These other frameworks include the situation where each player observes the actions played by the others, react to them by playing his BR, the others update their strategy accordingly, and so on. It turns out that these games can converge to an NE that would be obtained if the players were knowing the game completely and playing in one shot. Fig. 2 shows the possible NE in the PA game of [36] in two-band two-user interference relay channels. In this figure θ_i represents the power fraction user i allocates to a frequency band, $1-\theta_1$ being the fraction allocated to the other band. It can be checked that the sequence $\{\theta_i^{(0)}; \theta_{-i}^{(1)} = \mathrm{BR}_{-i}(\theta_i^{(0)}); \theta_i^{(2)} = \mathrm{BR}_i(\theta_{-i}^{(1)}); \theta_{-i}^{(3)} = \mathrm{BR}_{-i}(\theta_i^{(2)}), ...\}$ will converge to one of the three possible NE, depending on the *game starting point* i.e., on the value of $i \in \{1, 2\}$ and the value of $\theta_i^{(0)} \in [0, 1]$. As a more general conclusion, we see that the initial operating state of a network can determine the equilibrium state in decentralized networks having certain convergence properties. We cannot expand on this issue here but it is important to know that games with standard BRs, PGs, and SMGs have attractive convergence properties. For example, the authors of [51] have shown how simple learning procedures, based on mild information assumptions, converge to the NE predicted in the associated game with complete information.

More generally, if there is a certain hierarchy in the game, this hierarchy can be exploited by one or several players to enforce a given equilibrium. The desired equilibrium can be selected because of its *efficiency*. Therefore, equilibrium efficiency is also a way of selecting an equilibrium. The fact that this equilibrium will effectively occur depends on whether there exists an entity capable of influencing the game. In the scenario of [43] where two point-to-point communications compete with each other (interference channel), the network owner chooses the best location for the added relay in order to maximize the network sum-rate at the equilibrium.

VI. Efficiency

We have seen that a simple scenario where game theory is a natural paradigm is the case of decentralized wireless networks. A well-studied problem is the PC problem in the uplink of MACs. One of the motivations for decentralized PC in such a context is to decrease the amount of signaling sent by the BS for PC. But there is a priori no reason why a decentralized network of partially or totally autonomous and selfish terminals should perform as well as its centralized counterpart, for which the PC problem can be optimized jointly at the BS. This poses the problem of efficiency of the network. More specifically, for a decentralized network having a unique NE, it is important to characterize the equilibrium efficiency since it is the state at which the network will spontaneously operate. As mentioned above, if there are multiple equilibria, equilibrium efficiency can be used as a discriminant factor to select one of them. Whatever the equilibrium be unique or resulting of a selection, two critical issues arise: how to measure the network equilibrium efficiency (Sec. VI-A); how to improve its efficiency (Sec. VI-B) when the overall network performance at the equilibrium is found to be not satisfying.

A. Measuring equilibrium efficiency

A well-known way of characterizing efficiency of an equilibrium is to know whether it is Pareto-optimal (PO). Pareto-optimality is defined as follows.

Definition 16 (Pareto-optimality): A profile of strategies $\underline{s}^{\mathrm{PO}}$ *is said to be PO if there is a non-empty subset of players* $\mathcal{K}_{++}$ *such that*

$$\forall i \in \mathcal{K}_{++}, \forall \underline{s} \in \mathcal{S},\ u_i(\underline{s}^{\mathrm{PO}}) > u_i(\underline{s}), \tag{4}$$

and

$$\forall i \in \mathcal{K}/\mathcal{K}_{++}, \forall \underline{s} \in \mathcal{S},\ u_i(\underline{s}^{\mathrm{PO}}) \geq u_i(\underline{s}). \tag{5}$$

Said otherwise, $\underline{s}^{\mathrm{PO}}$ is Pareto-optimal if there exists no other profile of strategies for which one or more players can improve their utility without reducing the utilities of the others. A simple example of Pareto-optimal profile of strategies is as follows.

Example 17: Assume a canonical $2-$*user MAC [52][53],* $Y = X_1 + X_2 + Z$ *with a receiver implementing successive interference cancellation (SIC). Consider the very simple game where, knowing the decoding order* *used by the receiver, the users want to maximize their Shannon transmission rate* $u_i(\underline{p}) = \log_2(1 + \mathrm{SINR}_i)$ *by choosing the best value for their transmit power* $p_i \in [0, P_i^{\max}]$. *It can be checked that all the profiles of strategies corresponding to the full cooperation segment of the capacity region frontier are Pareto-optimal. In fact, along this segment, we even have a zero-sum game* $\forall \underline{p} \in [0, P_1^{\max}] \times [0, P_2^{\max}],\ u_1(\underline{p}) + u_2(\underline{p}) = R_{\mathrm{sum}}$, *where* $R_{\mathrm{sum}} = \log_2\left(1 + \frac{P_1^{\max}+P_2^{\max}}{\sigma^2}\right)$ *is the MAC sum-rate.*

In the example we have considered to illustrate Pareto-optimality we see that PO profiles of strategies are such that the sum of the utilities is maximized. This observation corresponds to the following general result (see e.g., [54]): every strategy profile $\underline{s}$ which maximizes the weighted sum $\sum_{i \in \mathcal{K}} \alpha_i u_i(\underline{s})$ is Pareto-optimal, with $\alpha_i > 0$. This result is useful to determine PO profiles of a game. In particular, the results holds when $\alpha_i = 1$. The corresponding function corresponds to a very well-known quantity: the social welfare. The social welfare of a game is defined as follows [55]:

Definition 18 (Social welfare): T*he social welfare of a game is defined as the sum of the utilities of all players:*

$$w = \sum_{i=1}^{K} u_i. \tag{6}$$

Social welfare, which corresponds to the average utility of the players (up to a scaling factor), is a well-known absolute measure of efficiency of a society, especially in economics. Is this quantity relevant in wireless communications? In theory, and more specifically in terms of ultimate performance limits of a network (Shannon theory), social welfare coincides with the network sum-rate. In contrast with many studies in economics, we have in communications, thanks to Shannon theory, a fundamental limit for the social welfare. For example, if we have K terminals, each of them implementing a selfish PC algorithm to optimize his Shannon transmission rate, communicating with two BSs connected with each other, we know that w cannot exceed the transmission rate of the equivalent virtual $K \times 2$ multiple input multiple output (MIMO) system. In practice, it can be a good measure if the players undergo quite similar propagation conditions, in which case, the utilities after averaging (e.g., over fading gains) can be close. If the users experience markedly different propagation conditions the use of social welfare can be sometimes arguable and even leads to very unfair solutions. We can mention at least three reasons why social welfare has to be replaced, in some scenarios, with other measures of global network performance. First, it is an absolute measure and therefore does not translate how large the gap between the performance of the decentralized network and its centralized counterpart is. Second, as mentioned, it can be unfair. Third, while it has a very nice physical interpretation when the users' utilities are chosen to be Shannon transmission rates, its meaning is much less clear in contexts where other utilities are considered (e.g., energy-efficiency). Before providing a way of dealing with the first drawback mentioned, let us illustrate the second and third ones by the example of energy-efficient PC games.

Example 19: In [41] the authors have studied, in particular, a non-cooperative energy-efficient PC game (see Ex. 3 for more details)) when the BS implements SIC. When the BS can optimize the decoding order associated with SIC, the authors have shown that after optimization of $w = \sum_{i=1}^{K} u_i = \sum_{i=1}^{K} \frac{f(\mathrm{SINR}_i)}{p_i}$*, the users who were "rich" in terms of link quality are now even "richer". This shows that* w *can be an unfair measure of energy-efficiency of the network. On other hand, the other performance metric considered in [41], the equivalent virtual MIMO system,* $v = \dfrac{\sum_{i=1}^{K} f(\mathrm{SINR}_i)}{\sum_{i=1}^{K} p_i}$*, is both fairer in terms of energy-efficiency and has a more physical interpretation than* w*.*

Let us go back to the first drawback of w. To deal with it, [56] introduced the concept of price of anarchy (PoA). The PoA was initially used to measure the inefficiency of equilibria in non-atomic selfish routing games, that is with a large number of players. In the context of games with a finite number of players, it is defined as follows.

Definition 20 (Price of anarchy): The PoA of the game $\mathcal{G}$ *is equal to the ratio of the highest value of the social welfare (joint optimization) to the worse NE of the game:*

$$\mathrm{PoA} = \frac{\max_{\underline{s} \in \mathcal{S}} w(\underline{s})}{\min_{\underline{s}^* \in \mathcal{S}^{\mathrm{NE}}} w(\underline{s}^*)} \tag{7}$$

where $\mathcal{S}^{\mathrm{SE}}$ *is set of Nash equilibria of the game.*

The price of stability (PoS) is defined similarly by replacing the denominator of the PoA with the best NE of the game, that is PoA and PoS coincide if there exists a unique NE. Interestingly, these quantities can be bounded in some cases. For instance, in the case of non-atomic routing games the PoA can be upper bounded by using variational inequalities. In a simple scenario where the cost functions of the players are affine functions this upper bound is $\frac{4}{3}$ (see e.g., [57]).

B. *Improving equilibrium efficiency*

When the performance of the network at the (considered) NE is found to be insufficient, the network or the game can be modified. There are many ways of doing this and we will just mention a few of them. What is important to have in mind is that the corresponding changes generally require allocating some resources (time-slots, band, ...) in order for the nodes to exchange some information and implement the new strategies.

A possible way of improving the equilibrium efficiency is to transform the non-cooperative game/network into a cooperative game/network. Note that we distinguish between cooperative networks and cooperative games. In a cooperative network, say a cooperative MAC where transmitters exchange cooperation signals [58], [59], the transmitters can still be selfish. In a cooperative game, some players help each other. For instance, in a *team game*, the players have a common objective and they do not necessarily exchange some signals for this purpose. A team formulation of the PC game introduced in [8] will however require full CSI at all the transmitters. Indeed, as mentioned in [41], if all the transmitters want to maximize w, every transmitter will need to know $h_1, ..., h_K$, in contrast with the non-cooperative game where only h_i is required at transmitter i. Of course, it is also possible and even required in a large network to form smaller groups of players: this is the principle of *coalitional games* (see e.g., [60], [61] for coalitional games with and without transferable utility respectively). If each member of a group or coalition has enough information, the coalition can even form a virtual antenna [62] and the gain brought by cooperation has to be shared between the players of the group. In such games, we see that we can have both locally cooperative networks and a non-cooperative game between the coalitions. In cooperative games, the counterpart of the NE solution concept for non-cooperative games is the Shapley value [63]. Note that there are other forms of cooperation, sometimes more implicit. This is the case of *repeated games* (see e.g., [64] for infinitely repeated games). Repeated games are a special case of dynamic games which consist in repeating at each step the same static game and the utilities results from averaging the static game utilities over time. In such a games, certain agreements between players on a common plan can be implemented, and a punishment policy can also be implemented to identify and punish the deviators. A simple wireless game where the concept of repeated games has been applied is the water-filling game by [39] to the fast fading MAC with single-antenna transmitters and a multi-antenna receiver. The authors show that the capacity region frontier is reached by repeating the non-cooperative game where selfish users maximize their transmission rate. At last we will mention an other form of cooperation: *bargaining*. Very interestingly, Nash has proved that cooperative games [60] can be studied with the same concepts as those used for non-cooperative games: a cooperative profile of strategies can be obtained as a subgame perfect equilibrium (which is an equilibrium having a certain robustness against changes in the main equilibrium plan, see [65] for more details) resulting from a bargaining procedure between the players. An example of application of this concept is [66] where two multi-antenna BSs cooperate by implementing a Nash bargaining solution. The bargaining solution has the advantage of being simple but for more than two players, this solution is rarely used because it does take into account the fact that an isolated group of players cooperating only with each other can be formed.

The ways of improving the efficiency of the network equilibrium we have mentioned so far, are generally very demanding in terms of CSI at the transmitters and can possibly require to establish new physical links between some nodes. Since cooperation can be costly in terms of additional resources, a more reasonable solution can be to merely coordinate the players. Here, we do not mean creating coordination games [46], which corresponds to a certain type of games we will not study here, but adding a certain degree of *coordination* in a non-cooperative game. Coordination between users can be stimulated, for instance, by using existing broadcasting signals like DVB or FM signals (case of public information), or by introducing dedicated signals sent by a BS (which can send both private and public information). The fact that all players of the game have access to certain (public or/and private) signals generally modify the players' behaviors. This knowledge can lead to a more efficient equilibrium, which can be a new NE or even a correlated equilibrium [14]. The authors of [67] have shown that, in a MAC where a slotted ALOHA protocol is assumed, a public coordination signal induces correlated equilibria that are more efficient in terms of reducing the frequency of collisions between users. More generally, it can be shown [14] that the set of achievable equilibria is enlarged by using common and private messages in the context of correlated equilibria. At last we will just mention two other usual techniques to improve the performance of a network at the equilibrium: (1) implementing a *pricing* technique or (2) introducing a certain degree of *hierarchy* in the game. Technique (1) has been used by [27] for energy-efficient PC games and by [39] for transmission rate-efficient PC games. Technique (2) has been used by [41] for energy-efficient PC games and by [68] for transmission rate-efficient PA games.

In this section we have mentioned several techniques to deal with equilibrium inefficiencies that are inherent to the non-cooperative game formulation. What is the best technique to be used? The answer to this question depends on many factors. Among the dominant factors we have the feasibility of the technique, predictability of the effective network state and of course the performance of the solution. Feasibility includes for example realistic information assumptions (CSI), complexity constraints at the terminals, problems of measurability. Predictability can be the impossibility to prove the uniqueness of the equilibrium. For example, implementing pricing necessitates to modify the original utility functions and uniqueness can be lost or not predictable after theses changes (see e.g., [27]). Performance can be a certain target in terms of quality of service.

VII. BEYOND NASH EQUILIBRIA IN STRATEGIC GAMES WITH FINITE NUMBER OF PLAYERS

In this article we have mainly focused on static/one-shot games with complete information and finite number of (rational) players. The methodology described in this article can be, to a large extent, re-used in other types of games having possibly different types of equilibrium. Taking into account the space constraints we had, the authors propose to briefly discuss two assumptions here: 1. The assumption on the number of players: What does the concept of NE become in games with a large number of players? An important concept of solution to these games is Wardrop equilibrium [69]; 2. The static game assumption: What does the concept of NE become in dynamic games?

A. Nash versus Wardrop in large games

To establish the link and differences between the Nash and Wardrop equilibria in a concrete way, our discussion is built on specific but useful scenarios in wireless communications, the problem of packets routing in wireless networks and ad hoc wireless networks.

Network games have been studied in the context of road traffic since the 1950s, when Wardrop proposed the following definition for a stable traffic flow on a transportation network: “The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route” (see p. 345 of [69]). In fact, this definition of equilibrium is different from the one proposed by Nash. Expressing the NE in terms of network flows, one can say that a network flow pattern is an NE if no individual decision maker in the network can change to a less costly strategy or route. When the number of decision makers in the game is finite, an NE can be achieved without the costs of all used routes being equal, contrary to Wardrop equilibrium (WE) principle. The WE assumes therefore that the contribution to costs or delays by any individual user is zero. In other words, the population of users is considered infinite. In some cases, Wardrop principle represents a limiting case of the NE principle as the number of users becomes very large [70]. Since routing in networks is also a fundamental part of communication networks, it was natural to expect the concepts of WE and of NE to appear in routing games. The natural analogy between packets in a network and cars on the road was not satisfactory for applying the Wardrop concept to networking, since packets, unlike cars, do not choose their routes. These are determined by the network routing protocols. Moreover, for a long time, routing issues did not concern wireless networking as it was restricted to the access part to a network where each terminal is associated with a given BS.

Routing has become relevant to wireless networking when ad-hoc networks have been introduced, see e.g., the pioneering work by Gupta and Kumar [71]. Yet game theory has penetrated into ad-hoc networks in a completely unexpected way. Indeed, to our knowledge, it is in this framework that non-cooperative game theory appears for the first time as a basic powerful tool to solve problems that do not involve competition. In [71], Gupta and Kumar propose a routing algorithm for packets that should satisfy some properties: packets should be routed by the network so as to follow shortest paths. It is thus not the average delay in the network that is minimized; the design objectives are to find routing strategies that result in (i) equalizing delays of packets that have the same source and destination so as to avoid re-sequencing delays that are quite harmful for real time traffic as well as for the TCP protocol for data transfers, and (ii) making these delays minimal, i.e., not to use routes that have delays larger than the minimal ones. The authors design a network routing protocol that achieves these two objectives. But these objectives are nothing else than the WE definition, a definition that arises in a completely non-competitive context. It is only ten years later that the WE penetrates into ad-hoc networks in a competitive game context. The problem of competitive routing in massively dense ad-hoc networks has been introduced in [72]. The author observes that as the density of mobiles grow, shortest paths tends to converge to some curves that can be described using tools from geometrical optics in non-homogeneous media (see Fig. 3). Later, the electromagnetic field has been used to approximate the routes of packets in massively dense ad-hoc networks (see [73] and references therein). Only later, in [74][75] the natural concept of WE was introduced in massively dense ad-hoc networks as a natural tool that deals with the competitive interactions between mobiles. This is a special version of WE defined on a continuum limit of a network, whose links and nodes are so dense that they are replaced by a plain.

B. Beyond static games

We have already mentioned in Sec. V the important fact that Nash equilibria predicted in a static game with complete information can be the effectively observed outcome of a game (with partial information) played in several steps where each player observes the actions played by the others, react to them by playing his BR, the others update their strategies accordingly, and so on. This clearly establishes a first link between static games and multi-step games. It turns out that the static game analysis is also very important to study a certain class of dynamic games namely repeated games, which consists, in their standard formulation [64], in repeating the same game at every time instance. The players want to maximize their utility averaged over the whole game duration. Interestingly, equilibria in certain repeated games can be predicted from the analysis of the game played in one shot. An important result in infinitely repeated games is the folk theorem, which provides the set of possible Nash equilibria of the repeated game (see e.g., [76] for more details).

Another important class of games are evolutionary games [13]. A concept of solution for these games is the concept of equilibria evolutionary stable (EES). This concept generalizes the NE in the sense that these equilibria are stable to the deviation of a fraction of the whole population (this therefore represents a large number). Additionally, these types of games are less demanding in terms of rationality assumptions.

VIII. Conclusion

Even in this short article, we have seen that game theory offers many strong results to prove existence of equilibria, select one of them when there are multiple equilibria and improve their efficiency. Note, however, that the current literature comprises not so many results to tackle the equilibrium uniqueness issue. In any case, all the provided results are based on certain assumptions in terms of information, rationality, and game stationarity. A real challenge for wireless games will be to be able to characterize equilibria under more realistic assumptions. For example, it is well known that CSI is always imperfect and that rationality can be an arguable assumption in heterogeneous wireless networks. And what about stationarity in wireless networks where channels can vary rapidly and the number of players can change? A real equilibrium theory specific to wireless networks will probably need to be built to address these issues. From the practical point of view, the learning approach in wireless networks opens a large avenue for technological innovation and designing algorithms implementable in real life wireless networks.

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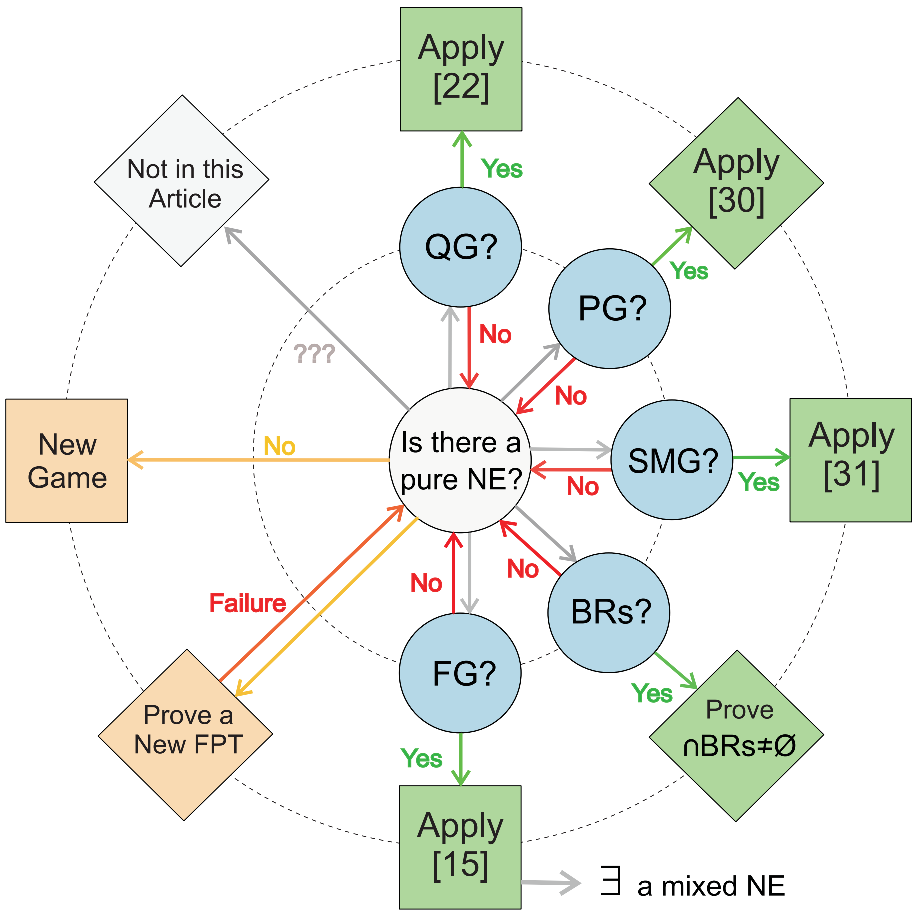

Fig. 1. A (non-exhaustive) methodology for proving the existence of a pure Nash equilibrium in strategic games. The reader has to refer to the acronyms and references used in Sec. III.

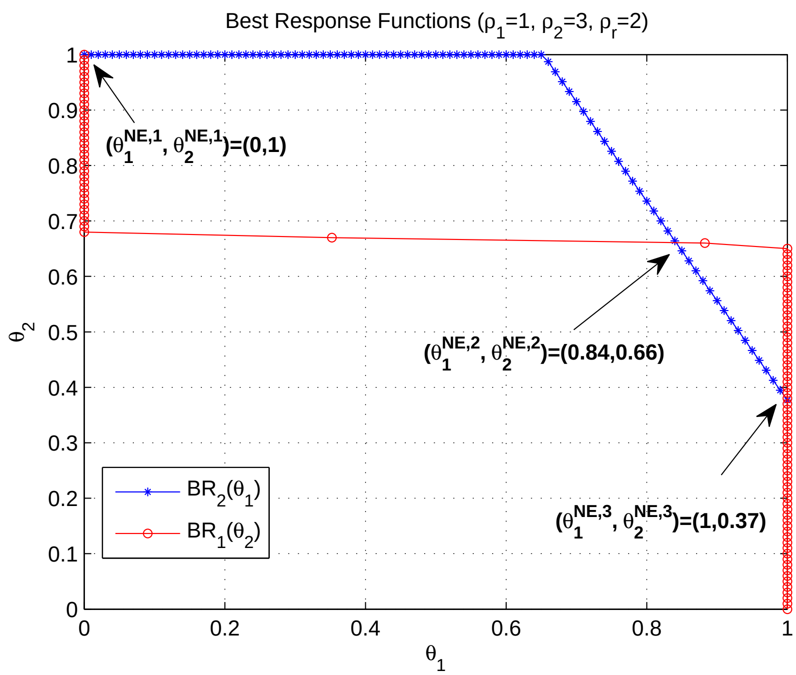

Fig. 2. In two-band two-user interference relay channels, PA games have multiple NE [36]. If the users plays and observe each other alternately, the game converges to an NE depending on the starting point of the game.

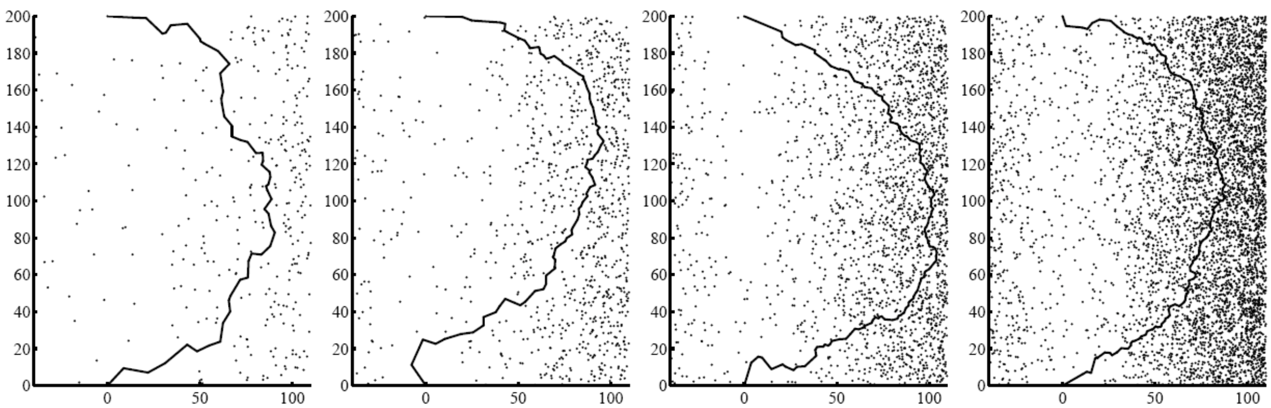

Fig. 3. Minimum cost routes in increasingly large networks. Optimal routes tend to follow to the Fermat principle for light propagation in non-homogeneous media.