Advantages of nonuniform arrays using Root-MUSIC

Carine El Kassis, José Picheral, Chafik Mokbel


Volume 90 Issue 2 February 2010 ISSN 0165-1684
ELSEVIER
SIGNAL
PROCESSING
An International Journal
A publication of the European Association for Signal Processing (EURASIP)
EURASIP

Signal Processing 90 (2010) 689–695

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Fast communication

Advantages of nonuniform arrays using root-MUSIC

Carine El Kassis [a,*], José Picheral [a], Chafic Mokbel [b]

[a] *Department of Signal Processing and Electronic Systems (SSE), SUPÉLEC, Plateau du moulon, 3, street Joliot-Curie, 91192 Gif-sur-Yvette Cedex, France*
[b] *University of Balamand, Deir El-Balamand, El-Koura, Lebanon*

ARTICLE INFO

Article history:
Received 8 April 2009
Accepted 14 July 2009
Available online 19 July 2009

Keywords:
Antenna arrays
Nonuniformly spaced arrays
Direction of arrival estimation
Root-MUSIC
Sparse array

ABSTRACT

In this paper, we consider the Direction-Of-Arrival (DOA) estimation problem in the Nonuniform Linear Arrays (NLA) case, particularly the arrays with missing sensors. We show that the root-MUSIC algorithm can be directly applied to this case and that it can fully exploit the advantages of using an NLA instead of a Uniform Linear Array (ULA). Using theoretical analysis and simulations, we demonstrate that employing an NLA with the same number of sensors as the ULA, yields better performance. Moreover, reducing the number of sensors while keeping the same array aperture as the ULA slightly modifies the Mean Square Error (MSE). Therefore, thanks to the NLA, it is possible to maintain a good resolution while decreasing the number of sensors. We also show that root-MUSIC presents good performance and is one of the simplest high resolution methods for this type of arrays. Closed-form expressions of the estimator variance and the Cramer–Rao Bound (CRB) are derived in order to support our simulation results. In addition, the analytical expression of the CRB of the NLA to the CRB of the ULA ratio is calculated in order to show the advantages of the NLA.

1. Introduction

The problem of estimating the Direction-Of-Arrival (DOA) has attracted a lot of attention in the last decades. In this paper, we consider the case of Nonuniform Linear Arrays (NLA), particularly the case of uniform grid arrays with missing sensors. In practice, some of the sensors in a uniform array may stop functioning, which yields an NLA. In this case, the array should be treated as nonuniform in order to optimize the DOAs estimator. Another application of NLA is the design of high performance and low cost arrays with reduced number of sensors. Reducing the number of sensors decreases the production cost as well as the computational time. This is due to the fact that nonregular geometry provides almost the same Mean Square Error (MSE) performance as the equivalent Uniform Linear Array (ULA).

It is well known that NLAs present sidelobe problems when using the classical beamforming algorithm for DOA estimation [1]. In this paper, we show that using High-Resolution (HR) methods (for instance root-MUSIC) overcomes this kind of problems. In particular, when using an NLA instead of a ULA, the MSE performance becomes slightly dependent on the number of sensors.

In the literature, many works have been reported to deal with the DOA estimation for NLAs. Thanks to its generality, spectral MUSIC [2] can be applied to any type of array geometry. Another class of methods involves applying some transformation to the measured data in order to obtain an interpolation of the data over a Virtual Uniform Linear Array (VULA). Consequently, the conventional methods can be applied to the interpolated data. Friedlander [3] proposes a sector-dependent interpolation followed by the conventional root-MUSIC. In [4], the authors propose the Expectation-Maximization (EM) algorithm in order to interpolate the observed data on a

* Corresponding author. Tel.: +33 1 69 85 14 21; fax: +33 1 69 85 14 29.
E-mail addresses: carine.elkassis@supelec.fr (C. El Kassis), jose.picheral@supelec.fr (J. Picheral), chafic.mokbel@balamand.edu.lb (C. Mokbel).

0165-1684/$ - see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.sigpro.2009.07.014

VULA using the noise-free model, followed by ESPRIT. Another method is proposed in [5], where the authors exploit the Toeplitz properties of the covariance matrix. In [6], a Higher Order Statistics (HOS) based method is proposed. Ref. [7] exploits the periodicity of spectral MUSIC criterion and uses the truncated Fourier series expansion of this function in order to transform the DOA estimation problem to a polynomial rooting problem. The authors in [8] model the NLA steering vector as a product of a matrix that depends only on the array parameters and a Vandermonde vector depending only on the angle. This Vandermonde structure is exploited to obtain a polynomial whose roots can be used to estimate the DOAs. The aforementioned methods suffer from being computationally expensive, or introduce errors due to the interpolation or to the truncation of the Fourier series expansion. In this paper, we use the simple root-MUSIC algorithm directly applied to the NLA in the case of arrays with missing sensors. Root-MUSIC is not restricted to the case of minimum redundant arrays, the missing sensors can be placed randomly in the array. This method does not require any additional data transformation and it can be considered as one of the simplest methods to deal with the NLA case. An analytic study of the variance of the DOA estimates is drawn in order to support our results.

Furthermore, it has been demonstrated in [9,10] that nonuniform spacing may lead to improved DOA estimation performance in terms of minimum variance. In this paper, we achieve an analytical and simulation performance study in order to show the advantages of using an NLA instead of a ULA for DOA estimation. Results show that using an NLA having the same aperture as a ULA but with significantly less number of sensors maintains good performance in MSE of the DOA estimates. In addition, we show that the performance of an NLA is better than the equivalent ULA with the same number of sensors, i.e. with a smaller aperture.

This letter is organized as follows. Section 2 introduces the signal model, the MUSIC and root-MUSIC algorithms for the DOA estimation in the case of nonuniform array geometries with missing sensors. Section 3 provides analytical expressions of the CRB and the variance of the root-MUSIC estimator of the DOA. In addition, a comparison is made with the CRB expression in the ULA case. Simulation results are presented in Section 4. In Section 5, an analysis showing the advantages of using nonuniform arrays instead of ULAs is achieved. Section 6 concludes our work.

2. MUSIC for NLA

2.1. Signal model

Consider N far-field narrowband sources incident on an M-element linear array, $(M \geq N)$, from directions $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_N]^\top$. The sensors, assumed to be omnidirectional, are situated at positions d_m $(m = 1, \ldots, M)$. We consider the case of arrays with missing sensors. As illustrated in Fig. 1, those NLAs can be considered as a ULA where some elements are omitted, i.e. $d_m = c_m \Delta$, where Δ is the ULA intersensor separation and c_m is an integer. Δ is taken as the half-wavelength $(\lambda/2)$ to avoid ambiguities.

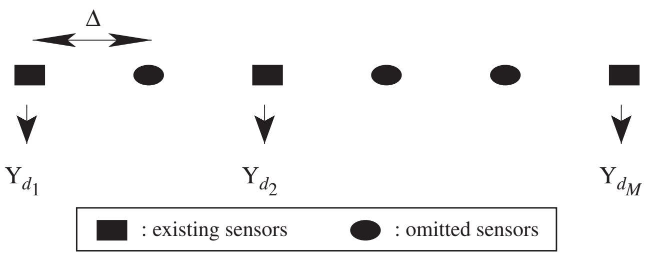

Fig. 1. Example of NLA and its equivalent ULA, with $\mathbf{d} = [0, 2, 5]\lambda/2$, i.e. $c_1 = 0$, $c_2 = 2$ and $c_3 = 5$.

By grouping the signals received by the M sensors in the $M \times 1$ vector $\mathbf{y}(t)$, the sensors output can be written as

$$\mathbf{y}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \boldsymbol{\nu}(t), \tag{1}$$

where $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_N)]$ is the $M \times N$ steering matrix and $\mathbf{a}(\theta_n)$ is the steering vector of the n-th source: $\mathbf{a}(\theta_n) = [e^{-j2\pi(d_1 \sin\theta_n/\lambda)}, \ldots, e^{-j2\pi(d_M \sin\theta_n/\lambda)}]^\top$. The $N \times 1$ vector $\mathbf{s}(t)$ contains the complex amplitude of the deterministic incident signals. As for the $M \times 1$ vector $\boldsymbol{\nu}(t)$, it represents a complex additive white Gaussian noise, with zero mean and a covariance $E\{\boldsymbol{\nu}(t)\boldsymbol{\nu}(t)^H\} = \sigma^2\mathbf{I}$. We assume that the sources are independent and that the received signal is sampled by L samples.

2.2. Spectral MUSIC

We briefly present in this paragraph the key idea of spectral MUSIC [2] that can be directly applied to NLA since the only assumption on the steering matrix $\mathbf{A}$ (or equivalently, on the array geometry) is that $Rank(\mathbf{A}) = N$. The covariance matrix of the observation vector $\mathbf{y}(t)$ is given by

$$\mathbf{R} = E\{\mathbf{y}(t)\mathbf{y}^H(t)\} = \mathbf{A}\mathbf{R_{ss}}\mathbf{A}^H + \sigma^2\mathbf{I}, \tag{2}$$

where $\mathbf{R_{ss}}$ is the covariance matrix of the source signals. Let $\mathbf{E}_N$ denote a basis of the noise subspace formed by the $(M-N)$ unit-norm eigenvectors associated with the $(M-N)$ smallest eigenvalues. It is straightforward that the noise subspace is orthogonal to the steering matrix and thus

$$\mathbf{a}^H(\theta_i)\mathbf{E}_N\mathbf{E}_N^H\mathbf{a}(\theta_i) = 0 \quad \text{for } i = 1, \ldots, N. \tag{3}$$

The solutions of (3) are the N DOAs $\{\theta_n\}_{n=1,\ldots,N}$ if and only if $\mathbf{A}\mathbf{R_{ss}}$ has a full rank, i.e. the columns of $\mathbf{A}$ are linearly independent and there are no correlated signals. The key idea of spectral MUSIC is to exploit the orthogonality property in (3). In practice, $\mathbf{R}$ is unknown, it is estimated using the available data: $\hat{\mathbf{R}} = (1/L)\sum_{i=1}^{L}\mathbf{y}(t_i)\mathbf{y}^H(t_i)$. Let $S_{MUSIC}(\theta) = 1/\mathbf{a}^H(\theta)\hat{\mathbf{E}}_N\hat{\mathbf{E}}_N^H\mathbf{a}(\theta)$ define the criterion of spectral MUSIC, where $\hat{\mathbf{E}}_N$ is an estimate of $\mathbf{E}_N$ based on the eigenvectors of $\hat{\mathbf{R}}$. The MUSIC estimates of θ_i are obtained by picking the N values of θ for which $S_{MUSIC}(\theta)$ is maximized.

2.3. Root-MUSIC

In this part, we introduce the root-MUSIC algorithm [11] applied to the case of NLA. Despite of what can be often read in literature, root-MUSIC is applicable not only in the ULA case, but also for the NLAs with missing sensors. The key idea is to exploit the orthogonality given by (3) in order to replace the search of the N maxima of the criterion by polynomial rooting. It is obvious that the true DOA values $\{\theta_n\}_{n=1}^{N}$ are the only solutions of (3). Since the nonuniform array can be considered as a ULA with missing sensors, the positions d_m $(m = 1, \ldots, M)$ of the existing sensors are given by $d_m = c_m \Delta$, where c_m is an integer. The steering vector can be written as: $\mathbf{a}(\theta_n) = [e^{-j2\pi(c_1 \Delta \sin\theta_n/\lambda)}, \ldots, e^{-j2\pi(c_M \Delta \sin\theta_n/\lambda)}]^\top$. Thus, let us define the $M \times 1$ vector $\mathbf{a}(z) = [z^{-c_1}, z^{-c_2}, \ldots, z^{-c_M}]^\top$, and the polynomial $G(z) = \mathbf{a}(z^{-1})^\top \hat{\mathbf{E}}_N \hat{\mathbf{E}}_N^H \mathbf{a}(z)$. Exploiting the orthogonality given by (3), it is easy to show that when $\hat{\mathbf{E}}_N = \mathbf{E}_N$ and $Rank(\mathbf{AR_{ss}}) = N$, the only $2N$ roots of $G(z)$ of unitary modulus have the form$\{z_i = e^{-j2\pi(\Delta \sin\theta_n/\lambda)}\}_{n=1,\ldots,N}$ and $\{\bar{z}_i = e^{j2\pi(\Delta \sin\theta_n/\lambda)}\}_{n=1,\ldots,N}$. Notice that the only difference with root-MUSIC for ULA is the way the polynomial $G(z)$ is formed. Afterwards, we follow the same steps of the original root-MUSIC algorithm [11]. Because of the presence of noise, the roots corresponding to the true DOA do not lie on the unit circle. Therefore, we choose the N roots with modulus nearest unity from among those lying inside the unit circle.

The root-MUSIC algorithm presents a lower computational cost in comparison to other HR methods for NLA.

3. Analytical study of the root-MUSIC performance

3.1. Root-MUSIC variance for NLA

In this paragraph, we formulate the variance of the proposed estimator. We follow the theoretical analysis proposed for spectral MUSIC in [12]. It has been shown in [13] that the variance of the DOA estimates obtained using spectral MUSIC or root-MUSIC takes the same expression. The variance of the DOA estimates is given by

$$var_{MUSIC}(\hat{\theta}_i) = \frac{\sigma^2}{2Lh(\theta_i)}\{[\mathbf{R}_{\mathbf{ss}}^{-1}]_{ii} + \sigma^2[\mathbf{R}_{\mathbf{ss}}^{-1}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{R}_{\mathbf{ss}}^{-1}]_{ii}\}, \tag{4}$$

where $h(\theta) = \mathbf{d}^H(\theta)[\mathbf{I} - \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H]\mathbf{d}(\theta)$, and $\mathbf{d}(\theta) = \partial\mathbf{a}(\theta)/\partial\theta$.

The expression of the root-MUSIC variance is also calculated in the case of one source $N = 1$:

$$var_{MUSIC_{NLA}}(\hat{\theta}) = \frac{\lambda^2}{8\pi^2\cos^2(\theta)M\sigma_d^2 SNR}\left[1 + \frac{1}{M \cdot SNR}\right], \tag{5}$$

where $\sigma_d^2 = (1/M)\sum_{m=1}^{M}(d_m - \bar{m}_d)^2$ is the variance matrix of the sensors distances d_m and SNR is the Signal-to-Noise Ratio. $\bar{m}_d = (1/M)\sum_{m=1}^{M} d_m$ is the mean of the sensors distances.

3.2. Cramer–Rao bound

In this section, we show the advantages of the NLA compared to the equivalent ULA, using the properties of the Cramer–Rao Bound (CRB). Let us denote by $\text{ULA}_{M'}$ the ULA with aperture and number of sensors equal to M' and by $\text{NLA}_{M',M}$ an NLA with aperture M' and M sensors. The expression of the CRB given in [13] can be generalized for the NLA case

$$CRB(\theta) = \frac{\sigma^2}{2}\left\{\sum_{t=1}^{L} \Re[\mathbf{X}^H(t)\mathbf{D}^H[\mathbf{I} - \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H]\mathbf{D}\mathbf{X}(t)]\right\}^{-1}, \tag{6}$$

where $\mathbf{X}(t) = \text{diag}[\mathbf{s}_1(t), \ldots, \mathbf{s}_N(t)]$ and $\mathbf{D} = [\mathbf{d}(\theta_1), \ldots, \mathbf{d}(\theta_N)]$. $\Re$ denotes the real part of the expression.

Considering the case of $N = 1$ source, (6) becomes

$$CRB(\theta) = \frac{\lambda^2}{8\pi^2\cos^2(\theta)M\sigma_d^2 SNR}. \tag{7}$$

Exploiting the variance properties, it is easy to show that for all the $\text{NLA}_{M',M}$:

$$\sigma^2_{d_{ULA_M}} \langle \sigma^2_{d_{NLA_{M',M}}} \Rightarrow CRB_{ULA_M} \rangle CRB_{NLA_{M',M}}. \tag{8}$$

The previous inequality means that in comparison to a ULA, an NLA with the same number of sensors M and a bigger aperture M', presents a lower CRB.

In the following, we consider the NLA with a centro-symmetric geometry (Fig. 2). It can be shown that this structure maximizes the σ_d^2 for a given number of sensors M and aperture M'. Thus, it is the optimal NLA structure for one source. The centro-symmetric geometry implies that the NLA has its missing sensors starting from the center of the array towards the extremum, as shown in Fig. 2. Exploiting the symmetry property of the geometry and $\bar{m}_d = 0$, we have

$$M\sigma_d^2 = 2\Delta^2 \sum_{m=(M'-M+1)/2}^{(M'-1)/2} d_m^2. \tag{9}$$

Let $\delta = M' - M$ be the number of missing sensors in the NLA. Skipping the calculation steps, the CRB takes the following expression:

$$CRB_{NLA_{M',M}} = \frac{3\lambda^2}{2\pi^2\Delta^2\cos^2(\theta)SNR[M'(M'^2 - 1) - \delta(\delta^2 - 1)]}. \tag{10}$$

Assuming that $M' \ll M'^3$ and $\delta \ll \delta^3$, the CRBs of the NLA and the ULA are approximated by

$$CRB_{NLA_{M',M}} \approx \frac{3\lambda^2}{2\pi^2\Delta^2\cos^2(\theta)SNR(M'^3 - \delta^3)}, \tag{11}$$

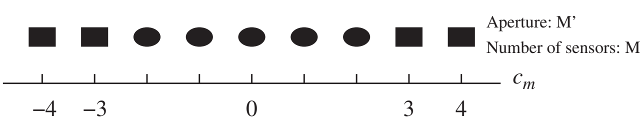

Fig. 2. Example of NLA with missing sensors starting from the center, with an aperture of $M' = 9$ and number of sensors $M = 4$.

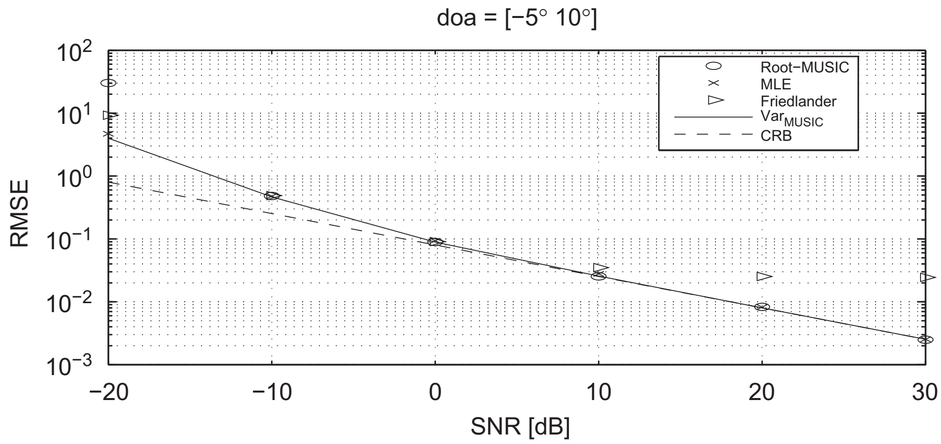

Fig. 3. RMS error for the source at 10° for the array $\mathbf{d}=[0,1,3,6,9]\lambda/2$, DOA $=[-5^\circ,10^\circ]$.

$$CRB_{ULA_{M'}} \approx \frac{3\lambda^2}{2\pi^2\Delta^2\cos^2(\theta)SNR(M')^3}. \tag{12}$$

Let us now compare a ULA and an NLA having the same aperture M', i.e. $\mathrm{ULA}_{M'}$ and $\mathrm{NLA}_{M',M}$. From (11) and (12), we obtain

$$\frac{CRB_{ULA_{M'}}}{CRB_{NLA_{M',M}}} \approx 1-\left(\frac{\delta}{M'}\right)^3. \tag{13}$$

If the number of missing sensors δ is small enough with respect to M', the last term in (13) is negligible. Thus, the NLA performance is almost the same as the ULA having the same aperture.

If we consider a ULA and an NLA having the same number of sensors, i.e. ULA_M and $\mathrm{NLA}_{M',M}$ as the case studied earlier, we have

$$\frac{CRB_{ULA_M}}{CRB_{NLA_{M',M}}} = 1+3\left(\frac{M'}{M}\right)^2-3\frac{M'}{M}>1. \tag{14}$$

Eq. (14) proves what has been said earlier in this paragraph: the NLA with a bigger aperture and the same number of sensors as the ULA presents better performance. The CRB ratio varies in a quadratic form with respect to M'/M. That means that the ULA performance degrades rapidly in comparison to the NLA when the aperture M' increases. These results are illustrated in Section 5.

4. Simulation results

Some simulations were conducted, in order to explore different aspects of root-MUSIC. The results are based on 500 trials in each case and $L=500$ snapshots are used. Consider the array defined by the sensors positions $\mathbf{d}=[0,1,3,6,9]\lambda/2$. This NLA presents a large number of omitted sensors and the intersensor separations have some missing lags. The narrowband signals are generated by two sources of equal power located at $[-5^\circ,10^\circ]$. The Root Mean Square Error (RMSE) of the DOA estimates is plotted with respect to the SNR.

Fig. 3 shows the simulation results for the first source ($\theta_1=10^\circ$). The results of the other source are not presented since the RMSE performance of both sources is equivalent. The RMSE of the proposed method is compared to the CRB and to the theoretical variance of the MUSIC estimator, given by (4). Furthermore, we compare root-MUSIC to the Maximum-Likelihood Estimator (MLE) and to the Friedlander interpolation method. We notice that the performance of root-MUSIC is very close to the performance of MLE, but root-MUSIC presents lower computational cost compared to the MLE. Furthermore, for high SNR, it is also very close to CRB, in contrast with the Friedlander interpolation estimator, which does not converge to the CRB. This is due to the fact that the Friedlander method introduces an interpolation error that does not decrease with the increase of the SNR or the snapshots number.

5. Advantages of NLA

In this section, we demonstrate the advantages of using the NLA instead of its equivalent ULA by applying root-MUSIC. In our simulations, we consider a ULA_{10} with $M'=10$ sensors spaced by a half-wavelength. The NLAs are created by eliminating the sensors successively, one after another. The choice of the NLA structures is made following the instruction in [10]: the optimal position estimation in an NLA is obtained by placing one-third of the sensors at each end and in the middle of the array. Similar results can be obtained when choosing other structures. The NLAs we have chosen are given in Table 1. Notice that all the $\mathrm{NLAs}_{10,M}$ have the same aperture as the original ULA_{10} with 10 sensors. The DOAs are located at $[-5^\circ,10^\circ]$. We apply the root-MUSIC algorithm to estimate the DOAs.

In the first experiment, we fix the SNR at 10 dB and we plot the RMSE versus the number of sensors. Fig. 4 shows the performance of the NLAs defined previously and the ULAs. For each value of M, we compare the performance of the array given by the corresponding row in Table 1. From Fig. 4, we see that $\mathrm{NLA}_{M',M}$ provides better performance than the ULA_M with the same number of sensors. Indeed, using an $\mathrm{NLA}_{10,5}$ with $M=5$ sensors and with an aperture of 10 gives better results than using a ULA_5 with five

Table 1
The NLAs and ULAs geometries used in the simulations.

Number of sensors (M)	NLA			ULA		
	Array	Aperture (M')	Positions [$\lambda/2$] (**d**)	Array	Aperture (M')	Positions [$\lambda/2$] (**d**)
10	–	–	–	ULA_{10}	10	[0 1 2 3 4 5 6 7 8 9]
9	$NLA_{10,9}$	10	[0 1 2 3 4 6 7 8 9]	ULA_9	9	[0 1 2 3 4 5 6 7 8]
8	$NLA_{10,8}$	10	[0 1 2 4 5 7 8 9]	ULA_8	8	[0 1 2 3 4 5 6 7]
7	$NLA_{10,7}$	10	[0 1 2 4 5 8 9]	ULA_7	7	[0 1 2 3 4 5 6]
6	$NLA_{10,6}$	10	[0 2 4 5 8 9]	ULA_6	6	[0 1 2 3 4 5]
5	$NLA_{10,5}$	10	[0 1 4 8 9]	ULA_5	5	[0 1 2 3 4]
4	$NLA_{10,4}$	10	[0 5 8 9]	ULA_4	4	[0 1 2 3]
3	$NLA_{10,3}$	10	[0 4 9]	ULA_3	3	[0 1 2]

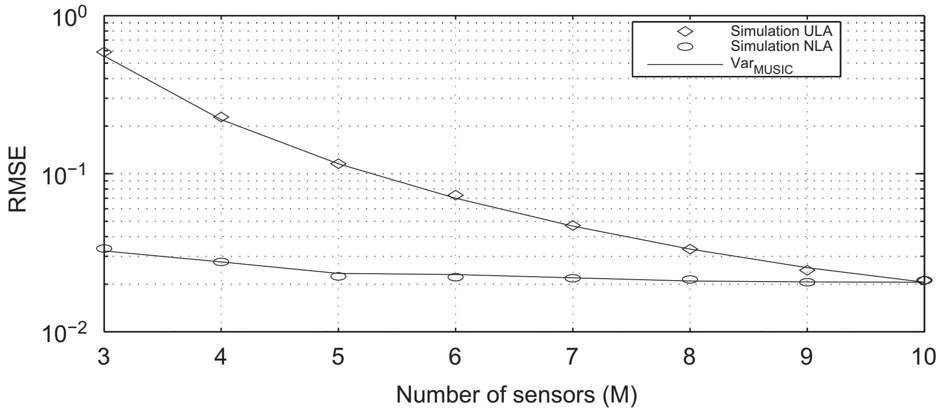

Fig. 4. RMS error for the source at -5° with $M = 3, 4, \ldots, 10$, SNR $= 10\,$dB, DOA $= [-5^\circ, 10^\circ]$ in the case of ULA and NLA.

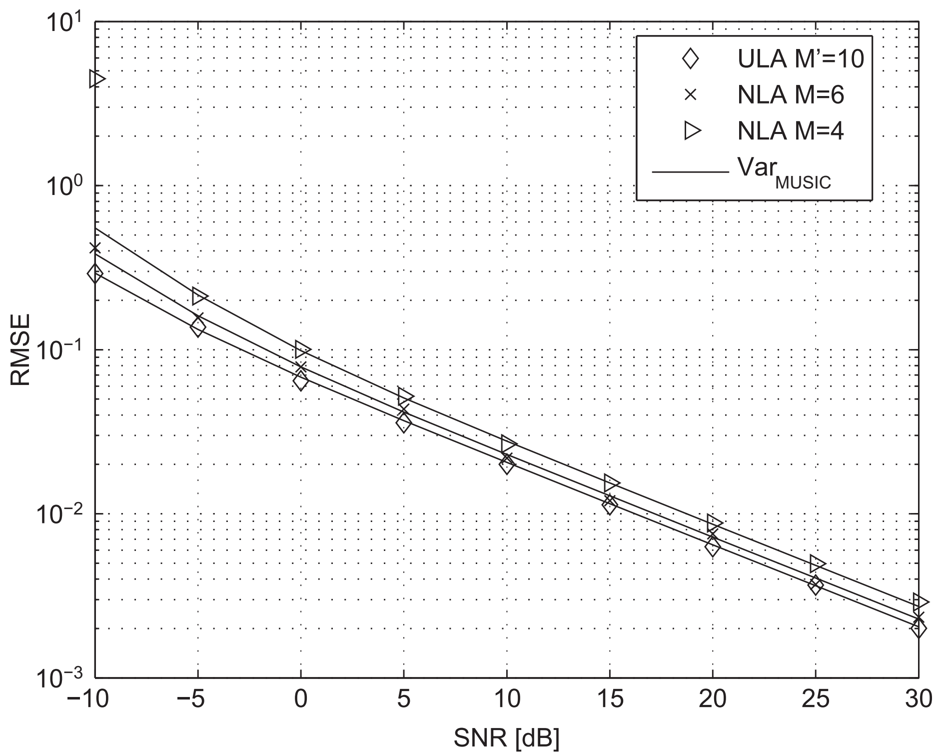

Fig. 5. RMS error for the source at -5° with $M' = 10$ in the case of ULA, $M = 6$ and 4 in the case of NLA, DOA $= [-5^\circ, 10^\circ]$.

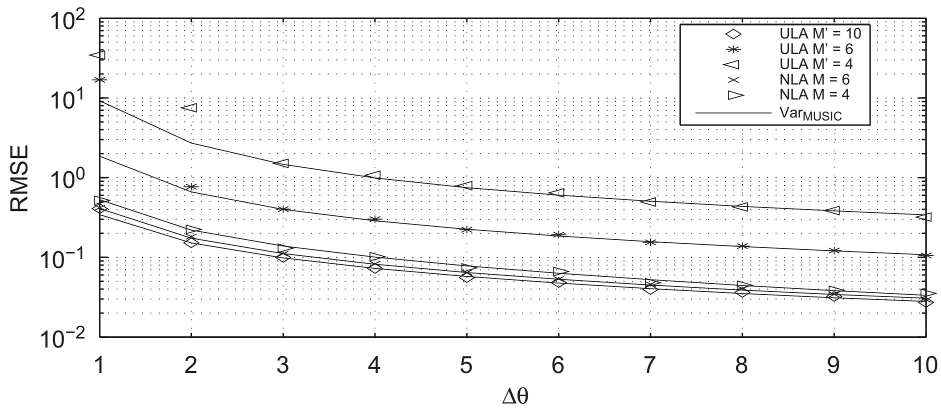

Fig. 6. Resolution for the source at 0° with $M' = 10$, 6 and 4 in the case of ULA, $M = 6$ and 4 in the case of NLA, at SNR $= 10\,\text{dB}$, $N = 2$ sources.

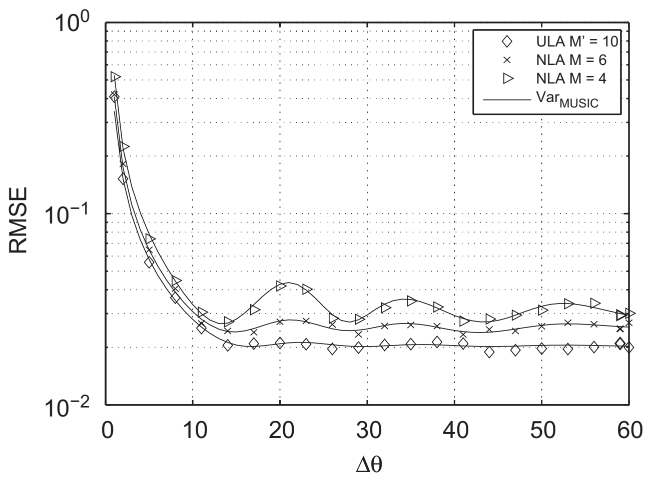

Fig. 7. RMS error for the source at 0° with ULA $M' = 10$ and NLAs with $M = 6$ and 4 respectively, at SNR $= 10\,\text{dB}$, $N = 2$ sources.

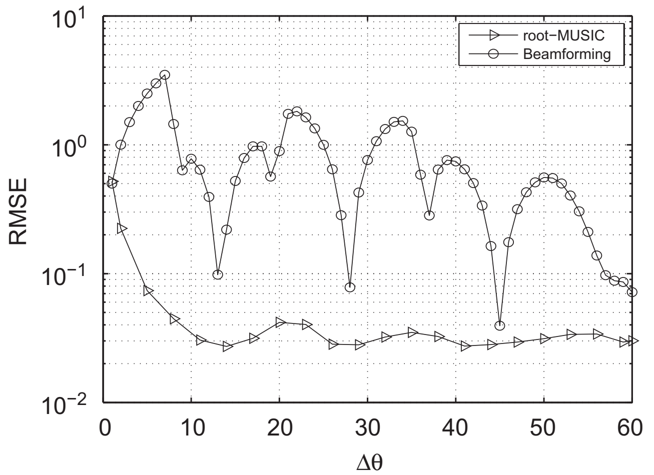

Fig. 8. RMS error for the source at 0° with NLA $M = 4$ using root-MUSIC and beamforming, at SNR $= 10\,\text{dB}$, $N = 2$ sources.

sensors. Thus, this example shows the importance of the aperture of the array and that root-MUSIC efficiently exploits the larger aperture of NLA with respect to ULA. (See closed expressions for $N = 1$ source in Section 3.)

Furthermore, if we focus on the NLA$_{10,M}$ curve, we can see that going from $M = 9$ to 4 slightly changes the performance. This means that instead of using 10 sensors, similar RMSE can be achieved using only half of the number of sensors, thanks to the HR methods like root-MUSIC. We conclude that the NLA may have numerous gaps without affecting the RMSE performance. Notice that using the standard beamforming with these array configurations cannot provide the same performance results due to the sidelobe problem.

In the next simulation, we emphasize the idea that using an NLA with a large number of omitted sensors slightly changes the performance. For that purpose, we take two of the NLAs$_{10,M}$ mentioned previously with $M = 6$ and 4, respectively, and compare them to the ULA$_{10}$ with $M' = 10$ sensors. Fig. 5 plots the evolution of the RMSE versus the SNR for the source at $-5°$. As we can see, the RMSE performance of the NLA$_{10,6}$ with $M = 6$ sensors is very close to the ULA$_{10}$ performance. When $M = 4$, the difference between the NLA$_{10,4}$ and the ULA$_{10}$ curve is less than 3 dB. Therefore, using an NLA having smaller number of sensors than the equivalent ULA can almost maintain the same RMSE performance.

In the third experiment, we evaluate the resolution of the root-MUSIC method for the same three arrays used above. We consider the case of two sources, where the fixed angle is 0° and the second angle is separated by $\Delta\theta$ increasing from 1° to 10°. The SNR is fixed to 10 dB. Fig. 6 shows the results for the source at 0°. We can see that root-MUSIC for NLA allows to obtain an angular resolution similar to the equivalent ULA (i.e. with the same aperture). On the other hand, if we compare the NLA to the ULA having the same number of sensors, we can see that the resolution in the NLA case is improved.

In the last experiment, we investigate the well known sidelobe issue due to the NLAs. We take the same case as above, i.e. the ULA$_{10}$ with $M' = 10$ sensors and the NLAs$_{10,M}$ with $M = 6$ and 4 sensors respectively. The SNR is fixed to 10 dB. We take the two sources case where the first angle is fixed to 0° and $\Delta\theta$ varies from 1° to 60°. In Fig. 7, we draw the RMSE of the three arrays mentioned above and in Fig. 8, we compare the RMSE of the NLA$_{10,4}$ with $M = 4$ sensors, using root-MUSIC and the classical beamforming. These figures show that using HR methods instead of beamforming can greatly reduce the effect of sidelobes. In fact, the sidelobe effect for the $NLA_{10,6}$ with $M=6$ sensors remains negligible. For the $NLA_{10,4}$ with $M=4$ sensors, this effect appears but remains minor with respect to the beamforming results.

6. Conclusion

In this paper, we have briefly presented the root-MUSIC algorithm directly applied to the nonuniform array case. This algorithm is simpler than many other methods for NLA that require an interpolation step or another complete treatment and presents good performance close to the MLE and the CRB. Its only limitation is that the NLA must be formed from a ULA with missing sensors. In addition, we have computed the variance of the DOA estimates for root-MUSIC in the NLA case and then compared analytically the performance of the NLA with respect to the equivalent ULA using the CRB expression. Simulation results show that root-MUSIC presents good performance for the NLA case. Furthermore, we emphasize that root-MUSIC can fully exploit the advantage of using an NLA instead of a ULA. In fact, for the same number of sensors, the NLA presents better performance since the aperture is bigger, which means that the resolution is better. With the same aperture, the performance is almost equivalent between the two array geometric types. This implies that using an NLA with a reasonable less number of sensors than the equivalent ULA preserves the same RMSE performance. Consequently, one of the advantages of a NLA lies in its economic aspect. These results were shown analytically and by simulations.

References

[1] H.R. Van Trees, Optimum Array Processing, Part IV of Detection, Estimation and Modulation Theory, Wiley, New York, 2002.
[2] R.O. Schmidt, A signal subspace approach to multiple emitter location and spectral estimation, Ph.D. Thesis, 1981.
[3] B. Friedlander, The root-MUSIC algorithm for direction finding with interpolated arrays, Signal Processing Journal 30 (1) (1993) 15–29.
[4] C. El Kassis, J. Picheral, C. Mokbel, EM-ESPRIT algorithm for direction finding with nonuniform arrays, in: Proceedings of the 14th Workshop on Statistical Signal Processing, Madison, WI, August 26–29, 2007, pp. 453–457.
[5] Y.I. Abramovich, D.A. Gray, A.Y. Gorokhov, N.K. Spencer, Positive-definite Toeplitz completion in DOA estimation for nonuniform linear antenna arrays—part I: fully augmentable arrays, IEEE Transactions on Signal Processing 46 (9) (1998) 2458–2471.
[6] Y.H. Chen, Y.S. Lin, Direction finding with nonuniform arrays via higher-order statistics, in: IEEE Antennas and Propagation Society International Symposium, vol. 3, 1994, pp. 1576–1579.
[7] M. Rübsamen, A.B. Gershman, Root-MUSIC direction-of-arrival estimation methods for arbitrary non-uniform arrays, in: Proceedings of the ICASSP, Las Vegas, NV, March 31–April 4, 2008, pp. 2317–2320.
[8] F. Belloni, A. Richter, V. Koivunen, DoA estimation via manifold separation for arbitrary array structures, IEEE Transactions on Signal Processing 55 (10) (2007) 4800–4810.
[9] G.C. Carter, Coherence and time delay estimation, Proceedings of the IEEE 75 (2) (1987) 236–255.
[10] A.J. Weiss, A.S. Willsky, B.C. Levy, Nonuniform array processing via the polynomial approach, IEEE Transactions on Aerospace and Electronic Systems 25 (1) (1989) 48–55.
[11] A.J. Barabell, Improving the resolution performance of eigenstructure-based direction-finding algorithm, in: Proceedings of the ICASSP, Boston, MA, 1983, pp. 336–339.
[12] P. Stoica, A. Nehorai, MUSIC, maximum likelihood and Cramer–Rao bound, IEEE Transactions on Acoustics, Speech and Signal Processing 37 (5) (1989) 720–741.
[13] B.D. Rao, K.V.S. Hari, Performance analysis of root-MUSIC, IEEE Transactions on Acoustics, Speech and Signal Processing 37 (12) (1989) 1939–1949.