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# ON THE ROBUSTNESS ANALYSIS OF TRIANGULAR NONLINEAR SYSTEMS: IISS AND PRACTICAL STABILITY

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## ABSTRACT

This note synthesizes recent results obtained by the authors on the stability and robustness analysis of cascaded systems. It focuses on two properties of interest when dealing with perturbed systems, namely integral input-to-state stability and practical stability. We present sufficient conditions for which each of these notions is preserved under cascade interconnection. The obtained conditions are of a structural nature, which makes their use particularly easy in practice.

**Index Terms**— Stability and robustness analysis, nonlinear systems, interconnected dynamical systems, cascades.

## 1. INTRODUCTION

The asymptotic stability analysis by Lyapunov's second method requires the construction of a strict Lyapunov function. This direct approach may be particularly hard for complex or large-scale nonlinear time-varying systems. A natural way of simplifying this problem consists in dividing the system into simpler interconnected subsystems, and to analyze each subsystem separately.

While a wide literature focuses on the feedback interconnection of dynamical subsystems, we concentrate here on recent results for the study of systems with a *triangular* or *cascade* form. Indeed, many applications can be represented as a *unidirectional* interconnection of dynamical subsystems.

Such a cascade structure often arises in the study of control systems. In particular, the so-called *cascades-based design* consists in designing a preliminary control law that makes the system have a cascade structure. This approach was followed in *e.g.* [6, 34] to control robot manipulators, where the mechanical (*i.e.* the robot arm) and the electrical (*i.e.* actuators) parts were addressed separately. See *e.g.* [33, 22, 25, 10, 18, 21, 44] for other examples of applications. Cascades *stabilization* was itself the subject of a great literature that we cannot cover here. We rather focus on the stability and robustness *analysis* of such systems. Indeed, in order to decompose a complex analysis

into simpler problems using theorems for cascaded systems, it is crucial to know whether the stability properties of both subsystems taken separately remain valid for their cascade interconnection.

From a theoretical point of view, this problem is not trivial. It has attracted the interest of the control community since [27], where graph theory was used to ensure local and global stability properties of the cascade, based on the assumption that the interconnection terms are all “stability preserving mappings”. In [50], converse Lyapunov results were used to show that uniform *local* asymptotic stability is naturally preserved by the cascade structure. Nevertheless, the global case presents harder difficulties. Intuitively, we could expect that, in order to preserve the global asymptotic stability of the cascade, it would suffice that the convergence rate of the driving subsystem be sufficiently high. This intuition is wrong in general, as proved in [45] through an elementary example involving a linear driving subsystem which yields a stronger peaking of the transients as the convergence is made faster. This transient peaking suffices to generate unbounded solutions. Similarly, as shown in [42, 47], neither integrability nor even exponential decay of the solutions of the driving subsystem is sufficient to preserve global asymptotic stability in general.

Beyond these obstacles, some sufficient conditions for the preservation of global asymptotic stability (GAS) under the cascade interconnection have been proposed in the literature. In general terms, a fundamental result for the analysis of global stability for nonlinear systems states that the cascade of GAS subsystems remains GAS if and only if its solutions are globally bounded. See [35, 38] for the proof of this statement in the case of autonomous systems and [32] for the case of time-varying systems.

Some work has then been done to advantageously replace the requirement of global boundedness of solutions by more structural conditions. In [35], these conditions take the form of a robustness Lyapunov condition on the driven subsystem that needs to hold for large values of the state. In [31, 26], uniform global boundedness of solutions is replaced by the requirements that the interconnection term be affine in the state of the driven subsystem, that the solutions of the driving subsystem be exponentially converging (or at least integrable) and that a Lyapunov func-

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The iISS results presented here were established by the first author in collaboration with D. Angeli, Università di Firenze, Italy.

tion, with a convenient bound on its gradient be known for the driven subsystem. In [32], further sufficient conditions are provided, expressed as dominance relationships linking the interconnection and drift terms. In [4], an elegant reformulation of the integrability condition posed in [31, 32] is established in terms of integral input-to-state stability (iISS<sup>1</sup>). More precisely, that reference proposes a condition linking the dissipation rate of the driving subsystem to the iISS gain of the driven under which the cascade interconnection of an iISS system driven by a GAS one is itself GAS. As iISS is a much weaker property than input-to-state stability (ISS) [39, 40], that result thus strongly relaxes the well known fact that a cascade of GAS subsystems is GAS if the driven subsystem is ISS, this fact being itself a direct consequence of the natural preservation of ISS under cascade interconnection [43]. The same results are highlighted from a different perspective in [31, 32].

Although very rich, most of this literature is concerned with the obtention of *asymptotic* stability properties for the overall cascade. However, when disturbances are considered (*e.g.* non-vanishing perturbations, model uncertainty or measurement imprecision), convergence to the origin may be impeded (steady-state error). This property is referred to as *ultimate boundedness*, *cf.* *e.g.* [14, 51]. In some practical circumstances, the operating point of a given system may be mathematically unstable, thus generating small oscillations around it, but still guarantee a sufficient precision for an acceptable behavior. This is especially true when feedback gains, or other free design parameters, may be tuned to arbitrarily reduce the amplitude of the steady-state error. This stronger property is referred to as *practical* stability. It will be the main subject of Section 3, in which we will recall the notion of global practical stability for nonlinear time-varying systems and present a growth rate condition, originally derived in [6], under which it is preserved by cascade interconnection.

The other series of results we recall in this note addresses a similar question for the iISS property. It is well known [4] that, contrarily to ISS, iISS is *not* preserved by cascade interconnection. We therefore provide conditions under which this preservation holds. These conditions are first given in the case when an explicit iISS Lyapunov-like function is known for each of the two subsystems. Roughly, it suffices that the dissipation term of the driving subsystem dominates the supply function of the driven one in a neighborhood of the origin. The second step consists in stating this condition in terms of the estimates of the trajectories of the two subsystems when disconnected. More precisely, in the case of a continuously differentiable zero-input driving subsystem, we impose, similarly to [4], that the driven subsystem presents a locally Lipschitz iISS gain, and the driving one be locally exponentially stable when no disturbance applies.

In addition, we complete the main result in [4] by giving a sufficient condition for the cascade composed of an iISS driven by a GAS one to remain GAS in the case when explicit Lyapunov functions are known. Roughly,

it is again required that the dissipation term of the GAS subsystem dominates the supply function of the iISS one around zero. This result may be useful in practice since the iISS and GAS properties are commonly established through Lyapunov arguments. Furthermore, this result naturally extends to multiple cascaded systems, *i.e.* series of cascaded iISS systems driven by a GAS one. These results, originally presented in [5], are all presented in Section 2.

**Notation:** Before continuing, we introduce the notation we will make use of along these lines.  $\mathcal{PD}$  denotes the class of all continuous positive definite functions  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ .  $\mathcal{K}$  designates the set of all continuous increasing functions  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  that vanish at 0. A function is said to belong to class  $\mathcal{K}_{\infty}$  if is of class  $\mathcal{K}$  and tends to infinity with its argument.  $\mathcal{L}$  is the class of all continuous decreasing functions  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  that tend to zero when their argument tends to infinity. A function is said to be in class  $\mathcal{KL}$  if it is of class  $\mathcal{K}$  in their first argument and of class  $\mathcal{L}$  in the second argument. Given any positive  $\varepsilon$ , the set of all functions  $\alpha$  satisfying  $\lambda\alpha(a+b) \leq \alpha(a) + \alpha(b)$  for all  $a, b \in (0, \varepsilon)$  and for some constant  $\lambda > 0$  is referred to as  $\mathcal{I}_{\varepsilon}$ . Given a positive  $\delta$ ,  $\mathcal{B}_{\delta}$  denotes the open ball of radius  $\delta$ , centered at the origin, in the Euclidean space of appropriate dimension. Let  $a \in \{0, +\infty\}$  and  $q_1$  and  $q_2$  be class  $\mathcal{K}$  functions. We say that  $q_1$  *dominates*  $q_2$  in a neighborhood of  $a$  (and we write  $q_2(s) = \mathcal{O}(q_1(s))$ ) if there exists a nonnegative constant  $k$  such that  $\limsup_{s \rightarrow a} q_2(s)/q_1(s) \leq k$ . We say that  $q_1$  *strictly dominates*  $q_2$  (notation:  $q_2(s) = o(q_1(s))$ ) if  $k$  can be taken as 0, and that  $q_1$  is *equivalent* to  $q_2$  (*i.e.*,  $q_1(s) \sim q_2(s)$ ) if  $\lim_{s \rightarrow a} q_2(s)/q_1(s) = 1$ .  $\llbracket a, b \rrbracket$  denotes the set of all integers in  $[a, b]$ . A dynamical system is said to be GAS if its origin is globally asymptotically stable. It is said to be LES if its origin is locally exponentially stable.

## 2. CASCADES INVOLVING IISS SYSTEMS

The results of this section, originally presented in [5], deal with nonlinear systems on which an input  $u$  is applied. We assume for simplicity that the only possible time-dependency occurs through the applied input. In other words, the class studied in this section encompasses all systems of the form

$$\dot{x} = f(x, u) \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state and  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a locally Lipschitz function. Input signals  $u$  are assumed to be in the class  $\mathcal{U}$ , made of all measurable locally essentially bounded functions  $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ .

The study of the impact of the input  $u$  on the overall behavior of the plant has been the subject of a rich literature. A particularly powerful framework to this respect is that of input-to-state stability (ISS) and its declinations; see [41] for a survey on this notion. The ISS property imposes that solutions be bounded by a fading function of the initial state plus a term that is somewhat “proportional” to the amplitude of the applied input. A generalization of this

<sup>1</sup>The definition of this property is recalled in the sequel.

property, involving a measure of the *energy* that the input feeds to the system (rather than its amplitude) is known as integral input-to-state stability (iISS). This notion happens to be a much wider concept than ISS and will be the key tool for the cascade analysis conducted here.

## 2.1. Related definitions

We start by recalling the definition of iISS itself and some properties that this concept naturally induces, as well as related notions.

**Definition 1 (iISS, [40])** *We say that (1) is integral input-to-state stable with respect to  $u$  if there exist functions  $\beta \in \mathcal{KL}$  and  $\gamma, \mu \in \mathcal{K}$  such that, for all  $x_0 \in \mathbb{R}^n$  and all  $u \in \mathcal{U}$ , its solution  $x(\cdot; x_0, u)$  satisfies, for all  $t \in \mathbb{R}_{\geq 0}$ ,*

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \gamma \left( \int_0^t \mu(|u(\tau)|) d\tau \right).$$

The function  $\mu$  is then referred to as an iISS gain for (1).

As established in [3], iISS is more conservative than requiring that the zero-input  $\dot{x} = f(x, 0)$  be GAS and that  $\dot{x} = f(x, u)$  be forward complete. Yet, it holds very often in specific applications involving GAS systems disturbed by external signals. Specific control design methods, such as [24], may also induce this property by state feedback.

While very general, this property induces interesting robustness properties on the perturbed system (1). For instance, it is shown in [40, Proposition 6] that, for any input  $u$  such that  $\int_0^\infty \mu(|u(\tau)|) d\tau$  is finite, the solution of an iISS system with iISS gain  $\mu$  eventually converges to zero.

It was established in [3] that a necessary and sufficient condition for a system like (1) to be iISS is that there exist a positive definite radially unbounded continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , a  $\mathcal{K}$  function  $\gamma$  and a  $\mathcal{PD}$  function  $\alpha$  satisfying, for all  $x \in \mathbb{R}^n$  and all  $u \in \mathbb{R}^m$ ,

$$\frac{\partial V}{\partial x}(x) f(x, u) \leq -\alpha(|x|) + \gamma(|u|).$$

This Lyapunov characterization is very similar to its ISS analogous, at the exception that  $\alpha$  is required to be a  $\mathcal{K}_\infty$  function for the latter.

We also recall that the system (1) is said to be 0-GAS (resp. 0-LES) if the origin of  $\dot{x} = f(x, 0)$  is GAS (resp. LES). It is said to satisfy the bounded energy - frequently bounded state (BEFBS) property if there exists  $\sigma \in \mathcal{K}_\infty$  such that, for all  $x_0 \in \mathbb{R}^n$ ,

$$\int_0^\infty \sigma(|u(\tau)|) d\tau < \infty \quad \Rightarrow \quad \liminf_{t \rightarrow +\infty} |x(t; x_0, u)| < \infty.$$

It was shown in [2] that the combination of 0-GAS and BEFBS is equivalent to iISS. This observation is at the basis of the proof of most results presented in this section.

## 2.2. Lyapunov-based condition for cascades composed of an iISS system driven by a GAS one

The first result we recall here is concerned with cascades composed of an iISS subsystem driven by a GAS one. It therefore focuses on systems of the form

$$\dot{x}_1 = f_1(x_1, x_2) \quad (2a)$$

$$\dot{x}_2 = f_2(x_2), \quad (2b)$$

where  $x_1 \in \mathbb{R}^{n_1}$ ,  $x_2 \in \mathbb{R}^{n_2}$  and the functions  $f_1$  and  $f_2$  are both assumed to be locally Lipschitz. We also assume that  $f_1(0, 0) = 0$  and  $f_2(0) = 0$ .

The first result we present here establishes a sufficient condition for a cascade of an iISS subsystem driven by a GAS one to be GAS: this conditions takes the form of an order comparison between the dissipation rate  $\alpha_2$  of the driving subsystem and the iISS gain  $\gamma_1$  of the driven one.

**Theorem 1 (GAS + iISS)** *Assume that the origin of (2b) is globally attractive<sup>2</sup> and that there exist a constant  $\varepsilon > 0$  and two continuous positive definite radially unbounded functions  $V_1$  and  $V_2$ , differentiable on  $\mathbb{R}^{n_1}$  and  $\mathcal{B}_\varepsilon \setminus \{0\}$  respectively, and satisfying*

$$\frac{\partial V_1}{\partial x_1} f_1(x_1, x_2) \leq -\alpha_1(|x_1|) + \gamma_1(|x_2|), \quad \forall (x_1, x_2) \in \mathbb{R}^{n_1+n_2} \quad (3)$$

$$\frac{\partial V_2}{\partial x_2} g(x_2) \leq -\alpha_2(|x_2|), \quad \forall x_2 \in \mathcal{B}_\varepsilon \setminus \{0\}, \quad (4)$$

where  $\alpha_1, \alpha_2 \in \mathcal{PD}$  and  $\gamma_1 \in \mathcal{K}$ . Assume further that

$$\gamma_1(s) = \mathcal{O}(\alpha_2(s)), \quad \text{as } s \rightarrow +\infty. \quad (5)$$

Then the cascade (2) is GAS.

Note that such a result, *i.e.* studying the cascade interconnection of an iISS subsystem driven by a GAS one, was also the purpose of [4]. The main difference here is that the GAS of the driving subsystem and the iISS of the driven one are not established through an explicit estimate of their solutions, but rather in terms of Lyapunov functions which constitute a natural tool for establishing these properties.

To this respect, note that the combination of global attractivity and (4) is actually equivalent to GAS of (2b). The above statement is motivated by an easier applicability in practice and the possibility it offers for generalizing to multiple cascades, as we will see later on in Theorem 2.

While authorizing a slight additional flexibility, as shown in the sequel, the fact of not requiring the differentiability of  $V_2$  at zero is motivated by homogeneity concerns with Theorem 4. The Lyapunov-like function constructed for the proof of the latter is indeed possibly non-differentiable at the origin.

A similar result can be derived from the recent contributions of H. Ito: [11, 12, 13]. In those references, a small gain condition for iISS systems is provided and, as a particular case, cascades composed of an iISS system

<sup>2</sup>*i.e.*, the solution of (2b) tends to the origin from any initial state.

driven by a GAS one are considered. The condition imposed there, however, involves the  $\mathcal{K}_\infty$  bounds on  $V_1$  and is expressed as a dominance condition on the whole state space, whereas the one presented here (5) needs to hold only in a neighborhood of the origin zero. Also, the conditions in those references implicitly require that the dissipation rate  $\alpha_2$  of the driving subsystem be of class  $\mathcal{K}$ , as this has been shown to be necessary for a general small gain theorem of not-necessarily ISS systems [1]. While those results apply to a more general context (feedback interconnection), these two features make the dominance condition (5) less conservative and easier to apply as far as cascades are concerned.

**Remark 1 (Relaxation of (5))** *It is worth mentioning that, if an upper bound on  $V_2$  of the form  $V_2(x_2) \leq \bar{\alpha}_2(|x_2|)$  is explicitly known, where  $\bar{\alpha}_2$  designates a  $\mathcal{K}_\infty$  function, condition (5) can be relaxed to the existence of a constant  $p \in [0, 1)$  such that*

$$\gamma_2(s) = \mathcal{O}\left(\frac{\alpha_2(s)}{\bar{\alpha}_2(s)^p}\right), \quad \alpha_2(s) = o(\bar{\alpha}_2(s)^p), \quad \text{as } s \rightarrow 0. \quad (6)$$

Indeed, consider the function  $\mathcal{V}_2(\cdot) := V_2(\cdot)^{1-p}$ . Then  $\mathcal{V}_2$  is a positive definite radially unbounded function, differentiable on  $\mathbb{R}^{n_2} \setminus \{0\}$ , and we get from (4) that

$$\begin{aligned} \frac{\partial \mathcal{V}_2}{\partial x_2}(x_2) f_2(x_2) &\leq -(1-p) \alpha_2(|x_2|) V_2^{-p}(x_2) \\ &\leq -(1-p) \frac{\alpha_2(|x_2|)}{\bar{\alpha}_2(|x_2|)^p} =: -\tilde{\alpha}_2(|x_2|). \end{aligned}$$

In view of (6),  $\tilde{\alpha}_2 \in \mathcal{PD}$ . Hence Theorem 1 applies with  $\mathcal{V}_2$ , and establishes that (2) is GAS. In this respect, notice that allowing  $V_2$  to be non-differentiable at the origin is useful, as further illustrated by the following example.

**Example 1** Consider the following two-dimensional system, consisting in a particular case of [4, Example 4]<sup>3</sup>:

$$\begin{aligned} \dot{x}_1 &= -\text{sat}(x_1) + x_1 x_2 \\ \dot{x}_2 &= -\frac{x_2}{1+x_2^2}, \end{aligned}$$

where  $\text{sat}(r) := \text{sign}(r) \min\{1, |r|\}$  for all  $r \in \mathbb{R}$ . Direct computations show that the functions  $V(x_1) = \frac{1}{2} \ln(1+x_1^2)$  and  $V_2(x_2) = \frac{1}{2} x_2^2$  satisfy (3) and (4) with  $\alpha_1(s) = (s \text{ sats}) / (1+s^2)$ ,  $\gamma_2(s) = s$  and  $\alpha_2(s) = s^2 / (1+s^2)$ . Since the requirement  $\gamma_2(s) = \mathcal{O}(\alpha_2(s))$  as  $s$  tends to zero does not hold, it is not possible to apply Theorem 1 directly. Nevertheless, it is possible to conclude GAS using the previous remark with  $p = 1/2$ . Indeed, take  $\bar{\alpha}(|x_2|) = \frac{1}{2} |x_2|^2 \geq V_2(x_2)$ , then  $\bar{\alpha}_2(s)^p = s / \sqrt{2}$  strictly dominates  $\alpha_2(s)$  around zero, and  $\alpha_2(s) / \bar{\alpha}_2(s)^p = s\sqrt{2} / (1+s^2)$  dominates  $\gamma_2(s)$  for small  $s$ .

We stress that, although the present use of Lyapunov-based conditions simplifies the argument, the above example can also be addressed by other methods such as [4, 32].

<sup>3</sup>It also appears in [32] and was originally suggested by Laurent Praly.

### 2.3. Lyapunov-based conditions for GAS of multiple systems in cascades

While Theorem 1 is stated for the cascade interconnection of two nonlinear subsystems, it easily extends to multiple cascades, *i.e.*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_{2 \rightarrow N}) \\ \dot{x}_2 &= f_2(x_2, x_{3 \rightarrow N}) \\ &\vdots \\ \dot{x}_{N-1} &= f_{N-1}(x_{N-1}, x_N) \\ \dot{x}_N &= f_N(x_N), \end{aligned} \quad (7)$$

where we used the notation  $x_{i \rightarrow j}$  to denote  $(x_i^T, \dots, x_j^T)^T \in \mathbb{R}^{n_i + \dots + n_j}$  for all integers  $0 \leq i \leq j$ . The functions  $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_i + \dots + N} \rightarrow \mathbb{R}^{n_i}$  are all assumed to be locally Lipschitz and to vanish at the origin.

The following result establishes GAS of (7) by assuming that<sup>4</sup> (7.N) is itself GAS and that all subsystems (7.i),  $i \in \llbracket 1, N-1 \rrbracket$ , are iISS. It relies on growth conditions (8) on the supply functions involved in the Lyapunov characterization of each subsystem involved. The proof requires a particular behavior of the involved supply functions in a neighborhood of zero. Namely, it imposes that they all belong to the class  $\mathcal{I}_\varepsilon$ , for some  $\varepsilon > 0$ , as defined in the notation section.

**Theorem 2 (GAS+iISS+iISS+...)** *Assume that the origin of (7.N) is globally attractive and that there exist a constant  $\varepsilon > 0$  and, for each  $i \in \llbracket 1, N \rrbracket$ , a continuous positive definite radially unbounded function  $V_i$ , differentiable on  $\mathbb{R}^{n_i}$ , and satisfying, for all  $i \in \llbracket 1, N-1 \rrbracket$ ,*

$$\frac{\partial V_i}{\partial x_i} f_i(x_i, x_{i+1 \rightarrow N}) \leq -\alpha_i(|x_i|) + \gamma_i(|x_{i+1 \rightarrow N}|), \quad \forall x_i \in \mathbb{R}^{n_i}$$

$$\frac{\partial V_N}{\partial x_N} f_N(x_N) \leq -\alpha_N(|x_N|), \quad \forall x_N \in \mathcal{B}_\varepsilon \setminus \{0\},$$

where, for each  $i \in \llbracket 1, N \rrbracket$ ,  $\alpha_i \in \mathcal{PD} \cap \mathcal{I}_\varepsilon$  and, for each  $i \in \llbracket 1, N-1 \rrbracket$ ,  $\gamma_i \in \mathcal{K} \cap \mathcal{I}_\varepsilon$ . Then, under the condition that

$$\gamma_i(s) = \mathcal{O}(\alpha_{i+1}(s)), \quad \forall i \in \llbracket 1, N-1 \rrbracket \quad (8a)$$

$$\gamma_i(s) = \mathcal{O}(\gamma_{i+1}(s)), \quad \forall i \in \llbracket 1, N-2 \rrbracket, \quad (8b)$$

as  $s$  tends to zero, the cascade (7) is GAS.

**Remark 2 (Differentiability of  $V_N$ )** *Although not stated explicitly for the sake of compactness,  $V_N$  is only required to be differentiable over  $\mathcal{B}_\varepsilon \setminus \{0\}$ .*

The proof of the above result is detailed in [5]. Note that the requirement that all supply functions be in the class  $\mathcal{I}_\varepsilon$  is little conservative in practice as it needs to hold only locally. For instance, it is satisfied by any function with polynomial behavior around zero.

<sup>4</sup>We use the notation (7.i) to denote the system  $\dot{x}_i = f_i(x_i, x_{i+1 \rightarrow N})$ .

#### 2.4. Lyapunov-based condition for cascades composed of two iISS systems

The next result studies the cascade connection of two iISS systems, in the case when an iISS-Lyapunov function is explicitly known for each of them. For the sake of generality, it is allowed that the driven subsystem depends also on the external input. We therefore deal with systems of the form:

$$\dot{x}_1 = f_1(x_1, x_2, u) \quad (9a)$$

$$\dot{x}_2 = f_2(x_2, u) \quad (9b)$$

where  $x_1 \in \mathbb{R}^{n_1}$ ,  $x_2 \in \mathbb{R}^{n_2}$ ,  $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$  is measurable locally essentially bounded,  $f_1$  and  $f_2$  are locally Lipschitz and satisfy  $f_1(0, 0, 0) = 0$  and  $f_2(0, 0) = 0$ .

**Theorem 3 (iISS+iISS)** *Let  $V_1$  and  $V_2$  be continuous positive definite radially unbounded functions, differentiable on  $\mathbb{R}^{n_1}$  and  $\mathbb{R}^{n_2}$  respectively. Suppose that there exist  $\nu_1 \in \mathcal{K}$  and, for all  $i \in \{1, 2\}$ ,  $\alpha_i \in \mathcal{PD}$  and  $\gamma_i \in \mathcal{K}$  such that, for all  $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  and all  $u \in \mathbb{R}^m$ ,*

$$\frac{\partial V_1}{\partial x_1} f_1(x_1, x_2, u) \leq -\alpha_1(|x_1|) + \gamma_1(|x_2|) + \nu_1(|u|) \quad (10)$$

$$x_2 \neq 0 \Rightarrow \frac{\partial V_2}{\partial x_2} f_2(x_2, u) \leq -\alpha_2(|x_2|) + \gamma_2(|u|). \quad (11)$$

Then the cascade (9) is iISS provided that

$$\gamma_1(s) = \mathcal{O}(\alpha_2(s)), \quad \text{as } s \rightarrow 0. \quad (12)$$

See [5] for the proof of this result. We stress that, unlike Theorem 1, the above result is not easily extendable to cascades involving more than two subsystems as condition (11) is required to hold on the whole  $\mathbb{R}^{n_2} \setminus \{0\}$  and not only locally as (4), which makes the dissipation rate resulting from the application of Theorem 3 difficult to estimate.

A direct consequence of Theorem 3, which is of notable interest in practice, concerns the case when the driven subsystem does not depend on the input  $u$ . The system then takes the more classical cascade form

$$\dot{x}_1 = f_1(x_1, x_2) \quad (13a)$$

$$\dot{x}_2 = f_2(x_2, u). \quad (13b)$$

**Corollary 1** *Let  $V_1$  and  $V_2$  be as in Theorem 3 and suppose that, for all  $i \in \{1, 2\}$ , there exist  $\alpha_i \in \mathcal{PD}$  and  $\gamma_i \in \mathcal{K}$  such that, for all  $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ , (11) holds and*

$$\frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1, x_2, u) \leq -\alpha_1(|x_1|) + \gamma_1(|x_2|).$$

Then the cascade (13) is iISS provided that (12) holds.

Intuitively, one could expect that the cascade inherits the iISS gain of its driving subsystem. Example 8 in [5] show that this intuition is not true in general.

#### 2.5. Trajectory-based condition for cascades of iISS systems

The last two results of this section propose a sufficient condition for a cascade of iISS systems to remain iISS, without requiring the knowledge of any Lyapunov function. Instead, greater stability properties are required for the subsystems involved. It is indeed imposed that the origin of the driving subsystem be locally exponentially stable when no input is applied, and that the iISS gain of the driven subsystem be locally Lipschitz. See [5, Section 4.1] for the proof.

**Theorem 4 (iISS + iISS, trajectory-based)** *Assume that  $f_2(\cdot, 0)$  is continuously differentiable. Assume that the system (9a) is iISS with respect to  $(x_2^T, u^T)^T$  with an locally Lipschitz iISS gain, and that the system (9b) is iISS and 0-LES. Then, the cascade (9) is iISS.*

It is worth noting that the condition we recover here is similar to the one derived from [4] for cascades constituted of an iISS subsystem driven by a GAS one. Condition (10) in [4] is indeed naturally fulfilled when the iISS gain is locally Lipschitz and the origin of the driving subsystem is locally exponentially stable.

Also, similarly to Theorem 3, note that this result applies to cascaded systems like (13), *i.e.* when the driven subsystem does not depend on  $u$ .

Under additional regularity conditions on the dynamics of the driven subsystems, the above result may extend to multiple cascaded systems, *i.e.*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \zeta_{2 \rightarrow N}) \\ \dot{x}_2 &= f_2(x_2, \zeta_{3 \rightarrow N}) \\ &\vdots \\ \dot{x}_{N-1} &= f_{N-1}(x_{N-1}, \zeta_{N \rightarrow N}) \\ \dot{x}_N &= f_N(x_N, u), \end{aligned} \quad (14)$$

where  $\zeta_{i \rightarrow N} := (x_i^T, \dots, x_N^T, u^T)^T \in \mathbb{R}^{n_i + \dots + n_N + m}$ , for all  $i \in \llbracket 2, N \rrbracket$ . By convention,  $\zeta_{N+1 \rightarrow N} := u$ . The following result was proved in [5, Section 4.5].

#### Theorem 5 (Cascade of multiple iISS systems)

*Assume that, for each  $i \in \llbracket 1, N \rrbracket$  the subsystem (14.i) is 0-LES and iISS with respect to  $\zeta_{i+1 \rightarrow N}$ . Assume also that, for each  $i \in \llbracket 1, N-1 \rrbracket$ , the function  $f_i(\cdot, 0)$  is continuously differentiable,  $\partial f_i(\cdot, 0)/\partial x_i$  is bounded in a neighborhood of the origin and the iISS gain of (14.i) is locally Lipschitz. Then the cascade (14) is iISS.*

### 3. GLOBAL PRACTICAL ASYMPTOTIC STABILITY

#### 3.1. Definition and Lyapunov characterization

The second part of this paper synthesizes recent results for the robustness analysis of cascades composed of parameterized nonlinear time-varying subsystems of the form

$$\dot{x} = f(t, x, \theta), \quad (15)$$

where  $x \in \mathbb{R}^n$  is the state,  $t \in \mathbb{R}_{\geq 0}$  is the time,  $\theta \in \mathbb{R}^m$  is a constant free parameter and  $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is locally Lipschitz in  $x$  and satisfies Carathéodory conditions for all  $\theta$  under consideration.

When (15) is a plant in closed-loop with a state feedback control law, the parameter  $\theta$  typically contains control gains (cf. e.g. [30, 7]). But  $\theta$  may also represent other design parameters cf. e.g. [49, 46, 48, 23, 20].

Contrarily to the previous section, we here consider the possible explicit time-dependency of the plant dynamics. While this time-dependency was implicitly contained in the input  $u$  applied to the systems studied in Section 2, the present section does not explicitly consider these inputs. The time-dependency of (15) may arise from the application of such an exogenous signal, but it may result from other causes such as trajectory tracking, control of plants that are not stabilizable by continuous time-invariant feedback<sup>5</sup> or systems with time-varying parameters. While the tools we present in the sequel allow consequently for a time-dependency of the Lyapunov functions involved, the properties we derive are all *uniform* in the initial time, meaning that the overall behavior of the solutions of (15) is independent of the initial time.

**Definition 2 (UGPAS)** Let  $\Theta \subset \mathbb{R}^m$  be a set of parameters. The system (15) is said to be uniformly globally practically asymptotically stable on  $\Theta$  if, given any positive  $\delta$ , there exists  $\theta^*(\delta) \in \Theta$  and  $\beta_\delta \in \mathcal{KL}$  such that, for any  $x_0 \in \mathbb{R}^n$  and any  $t_0 \in \mathbb{R}_{\geq 0}$ , the solution of  $\dot{x} = f(t, x, \theta^*)$  satisfies

$$|x(t; t_0, x_0, \theta^*)| \leq \delta + \beta_\delta(|x_0|, t - t_0), \quad \forall t \geq t_0.$$

Hence, roughly speaking, we say that (15) is UGPAS if the size of the ball to which solutions converge can be arbitrarily diminished by a convenient choice of  $\theta$ . In other words, although asymptotic convergence to zero may be impeded<sup>6</sup>, the precision reached by the plant can be made arbitrarily tight by the proper use of control parameters. Such a situation is fairly common in control practice, notably in the case of UGAS controlled systems perturbed by bounded external disturbances.

The above notion of practical stability should not be confused with existing definitions in the literature that correspond more to the concept of *ultimate boundedness* (see e.g. [14]), in the sense that they require that solutions eventually enter a ball without leaving it anymore but do not impose that this ball be reducible at will by tuning a parameter.

Other definitions are more conservative than Definition 2, as they require that the  $\mathcal{KL}$  estimate, or at least its dependency on the initial state, be uniform in the parameters  $\theta \in \Theta$  (not allowing  $\beta$  to depend on the precision  $\delta$  one wants to reach). While the latter property is satisfied in many contexts (see e.g. [49, 28, 46]), it may fail when

<sup>5</sup>It was shown in [9] that any completely controllable smooth system without drift (including nonholonomic mechanical systems) can be stabilized by means of a smooth periodic time-varying state feedback.

<sup>6</sup>The origin may not even be an equilibrium.

dealing with perturbed systems: the transients overshoot being often linked to the control gains (see [7] for an example in robot control). It should therefore be clear that, in Definition 2, “uniform” refers only to the initial conditions, and not to the tuning parameter.

Finally, we stress that other definitions, such as in [49, 29], require that the tuning parameter be a positive scalar that needs to be diminished in order to get a better precision, in which case any smaller parameter is guaranteed to induce at least the same precision. No such tuning procedure is imposed by Definition 2 as the parameter  $\theta$  does not need to be scalar, making possible to take into account multiple gains tuning (see e.g. [8, 19, 16] for applications involving mechanical systems).

The following result, proved in [6, Section IV-A], gives a sufficient condition, in terms of a Lyapunov function defined out of a neighborhood of the origin, for the dynamical system (15) to be uniformly globally practically asymptotically stable on a given set of parameters.

**Proposition 1 (Lyapunov condition for UGPAS)**

Suppose that, given any  $\delta > 0$ , there exist a parameter  $\theta(\delta) \in \Theta$ , a continuously differentiable Lyapunov function  $V_\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}_\delta, \bar{\alpha}_\delta, \alpha_\delta$  such that, for all  $x \in \mathbb{R}^n \setminus \mathcal{B}_\delta$  and all  $t \in \mathbb{R}_{\geq 0}$ ,

$$\underline{\alpha}_\delta(|x|) \leq V_\delta(t, x) \leq \bar{\alpha}_\delta(|x|) \quad (16)$$

$$\frac{\partial V_\delta}{\partial t}(t, x) + \frac{\partial V_\delta}{\partial x}(t, x)f(t, x, \theta) \leq -\alpha_\delta(|x|). \quad (17)$$

Then the system (15) is UGPAS on the parameter set  $\Theta$  provided that the following relation holds:

$$\lim_{\delta \rightarrow 0} \underline{\alpha}_\delta^{-1} \circ \bar{\alpha}_\delta(\delta) = 0. \quad (18)$$

Compared to classical results for Lyapunov stability, conditions (16) and (17) are natural: they are similar to the Lyapunov sufficient condition for global ultimate boundedness (cf. e.g. [14]). Intuitively, one may expect that these two requirements, when valid for any arbitrarily small tolerance  $\delta$ , suffice to conclude UGPAS. However, an additional assumption (18) is required that links the bounds on the Lyapunov function. Indeed, as opposed to previously cited definitions of practical stability, the Lyapunov function may depend on the tuning parameter  $\theta$ , and consequently on the desired precision  $\delta$ . As clearly shown in [36, 15], this parametrization of the Lyapunov function may induce unexpected behaviors of solutions if (18) is violated.

We finally stress that, although not included in the present paper, similar results as those presented here were derived for *semiglobal* stability. This situation refers to the case when the domain of attraction is not the whole state space but rather a compact neighborhood of the origin whose size can be enlarged at will by conveniently tuning the control parameters. We refer the reader to [7] for details on these aspects.

In many applications, the requirement (18) that links the lower and upper  $\mathcal{K}_\infty$  bounds on the Lyapunov function

follows from the combination of three properties: these bounds are affine in the tuning parameters  $\theta$ , they are polynomial functions of the same degree, and the parameters are affine in the inverse of the radius  $\delta$  of the attractive ball. We therefore recall the following result that especially fits this situation. Although less general, it is more easily applicable. See [16, 19] for applications of this corollary in control of a spacecraft formation and for the automatic positioning of ships for underway replenishment.

**Corollary 2 (Simplified condition for UGPAS)**

Let  $\Theta$  be a subset of  $\mathbb{R}^n$ . Assume that there exist a positive constant  $p$ , real constants  $\underline{a}_i, \bar{a}_i, \underline{b}_i, \bar{b}_i, i \in \llbracket 1, n \rrbracket$  and, for any  $\theta \in \Theta$ , a continuously differentiable Lyapunov function  $V_\theta$  satisfying, for all  $x \in \mathbb{R}^n$  and all  $t \in \mathbb{R}_{\geq 0}$ ,

$$\sum_{i=1}^n (\underline{a}_i + \underline{b}_i \theta_i) |x_i|^p \leq V_\theta(t, x) \leq \sum_{i=1}^n (\bar{a}_i + \bar{b}_i \theta_i) |x_i|^p \quad (19)$$

where, for all  $i \in \llbracket 1, n \rrbracket$  and all  $\theta \in \Theta$ ,  $\underline{a}_i + \underline{b}_i \theta_i > 0$  and  $\bar{a}_i + \bar{b}_i \theta_i > 0$ . Suppose further that, given any  $\delta > 0$ , there exist a parameter  $\theta^*(\delta) \in \Theta$  and a class  $\mathcal{K}_\infty$  function  $\alpha_\delta$  such that, for all  $x$  such that  $|x| \geq \delta$  and all  $t \in \mathbb{R}_{\geq 0}$ ,

$$\frac{\partial V_{\theta^*}}{\partial t}(t, x) + \frac{\partial V_{\theta^*}}{\partial x}(t, x) f(t, x, \theta^*) \leq -\alpha_\delta(|x|). \quad (20)$$

If, furthermore, for all  $i \in \llbracket 1, n \rrbracket$ , it holds that

$$\lim_{\delta \rightarrow 0} \underline{a}_i + \underline{b}_i \theta_i^*(\delta) > 0, \quad (21)$$

$$\bar{b}_i \neq 0 \quad \Rightarrow \quad \lim_{\delta \rightarrow 0} \theta_i^*(\delta) \delta^p = 0, \quad (22)$$

then the system (15) is UGPAS on the parameter set  $\Theta$ .

**Remark 3 (High gain)** When  $\theta$  represents control gains, these usually need to be enlarged in order to achieve a better precision, which makes (21) satisfied in most cases.

**Remark 4 (Quadratic Lyapunov function)** It should be stressed that (19) holds in particular for quadratic Lyapunov function  $V_\theta(x) = x^T P(\theta)x$  where  $P \in \mathbb{R}^{n \times n}$  can be decomposed as  $P(\theta) = P_1 + P_2 \theta$ , where  $P_1, P_2 \in \mathbb{R}^{n \times n}$  are independent of  $\theta$ .

**3.2. Cascades of UGPAS systems**

Based on these preliminaries about UGPAS, we now present recent results for the study of cascade interconnections of UGPAS systems. More precisely, we consider cascaded systems of the form

$$\dot{x}_1 = f_1(t, x_1, \theta_1) + g(t, x, \theta) \quad (23a)$$

$$\dot{x}_2 = f_2(t, x_2, \theta_2) \quad (23b)$$

where  $x := (x_1^T, x_2^T)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ ,  $\theta := (\theta_1^T, \theta_2^T)^T \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ ,  $t \in \mathbb{R}_{\geq 0}$ ,  $f_1, f_2$  and  $g$  are locally Lipschitz in the state and satisfy the Carathéodory conditions for all considered  $\theta$ . Note that, compared to the results of Section 2, the cascade structure is slightly more constrained as it

imposes that the interconnection term  $g(t, x, \theta)$  appears as an additive term in the nominal dynamics of the driven subsystem  $\dot{x}_1 = f_1(t, x_1, \theta_1)$ . We further assume that this interconnection term is uniformly bounded both in time and in  $\theta_2$  and vanishes with  $x_2$ :

**Assumption 1 (Bound on the interconnection term)**

For any  $\theta_1 \in \Theta_1$ , there exists a nondecreasing function  $G_{\theta_1}$  and a class  $\mathcal{K}$  function  $\Psi_{\theta_1}$  such that, for all  $\theta_2 \in \Theta_2$ , all  $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  and all  $t \in \mathbb{R}_{\geq 0}$ ,

$$|g(t, x, \theta)| \leq G_{\theta_1}(|x|) \Psi_{\theta_1}(|x_2|).$$

We also assume that each subsystem is UGPAS when considered independently, and that we explicitly know a Lyapunov function characterizing this property for the driven subsystem.

**Assumption 2 (UGPAS driving)** The driving subsystem (23b) is UGPAS on  $\Theta_2$ .

**Assumption 3 (UGPAS driven + growth restrictions)**

Given any  $\delta_1 > 0$ , there exist a parameter  $\theta_1^*(\delta_1) \in \Theta_1$ , a continuously differentiable Lyapunov function  $V_{\delta_1}$ , class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}_{\delta_1}, \bar{\alpha}_{\delta_1}, \alpha_{\delta_1}$  and a continuous positive nondecreasing function  $c_{\delta_1}$  such that, for all  $x_1 \in \mathbb{R}^{n_1} \setminus \mathcal{B}_{\delta_1}$  and all  $t \in \mathbb{R}_{\geq 0}$ ,

$$\underline{\alpha}_{\delta_1}(|x_1|) \leq V_{\delta_1}(t, x_1) \leq \bar{\alpha}_{\delta_1}(|x_1|) \quad (24)$$

$$\frac{\partial V_{\delta_1}}{\partial t} + \frac{\partial V_{\delta_1}}{\partial x_1} f_1(t, x_1, \theta_1^*) \leq -\alpha_{\delta_1}(|x_1|) \quad (25)$$

$$\left| \frac{\partial V_{\delta_1}}{\partial x_1}(t, x_1) \right| \leq c_{\delta_1}(|x_1|) \quad (26)$$

$$\lim_{\delta_1 \rightarrow 0} \underline{\alpha}_{\delta_1}^{-1} \circ \bar{\alpha}_{\delta_1}(\delta_1) = 0. \quad (27)$$

In addition, for the function  $G_{\theta_1}$  of Assumption 1, it holds that, for each  $\delta_1 > 0$  and as  $s$  tends to  $+\infty$ ,

$$c_{\delta_1}(s) G_{\theta_1^*(\delta_1)}(s) = \mathcal{O}(\alpha_{\delta_1} \circ \bar{\alpha}_{\delta_1}^{-1} \circ \underline{\alpha}_{\delta_1}(s)) \quad (28a)$$

$$\alpha_{\delta_1}(s) = \mathcal{O}(\bar{\alpha}_{\delta_1}(s)). \quad (28b)$$

The specific assumptions (26) and (28) made on the gradient of the Lyapunov function for the driven subsystem (and thus on the growth of the interconnection term) actually guarantee that the solutions of the overall cascade remain bounded, which is a key issue in cascade reasoning [35, 32, 37]. Note that this strong feature is ensured based on algebraic considerations, which simplifies the analysis.

Based on these three assumptions, we state the following result, originally proved in [6].

**Theorem 6 (UGPAS + UGPAS)** Under Assumptions 1, 2 and 3, the cascaded system (23) is UGPAS on the parameter set  $\Theta_1 \times \Theta_2$ .

This result was successfully applied to spacecraft formation control [17], underway ship replenishment [19], PID control of robot manipulators [8] and disturbance rejection by smooth control [6].

As Theorem 6 does not provide information on the Lyapunov function associated to the cascade (23), it was not yet possible to extend this result to multiple cascaded systems such as in Section 2.3.

#### 4. CONCLUDING REMARKS

This paper has synthesized different recent results obtained by the authors in the field of robustness analysis of cascaded systems. Two different approaches were considered.

The first one aims at considering explicitly the effect of a disturbance  $u$  on the overall behavior of cascaded systems, using the formalism of iISS. More precisely, sufficient conditions were provided under which a cascade composed of an iISS subsystem driven by a GAS one is itself GAS. Similar developments were made for cascades of iISS subsystems.

The second approach was concerned with the practical stability that can be guaranteed to cascaded subsystems when free parameters can be tuned in order to reject the effect of perturbations.

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