Spectrum Sharing Games on the Interference Channel

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***Abstract*—In this paper, we address the problem of spectrum sharing where competitive operators coexist in the same frequency band. First, we model this problem as a strategic non-cooperative game where operators *simultaneously* share the spectrum according to the Nash Equilibrium (N.E). Given a set of channel realizations, several Nash equilibria exist which renders the outcome of the game unpredictable. For this reason, the spectrum sharing problem is reformulated as a *Stackelberg* game where the first operator is already being deployed and the secondary operator follows next. The Stackelberg equilibrium (S.E) is reached where the best response of the secondary operator is taken into account upon maximizing the primary operator's utility function. Finally, we assess the goodness of the proposed distributed approach by comparing its performance to the centralized approach.**

I. Introduction

Spectrum sharing between wireless networks improves the efficiency of spectrum usage, and thereby alleviates spectrum scarcity due to growing demands for wireless broadband access. To improve the under-utilization of spectrum resources, we study spectrum sharing between two competing operators operating in the same frequency band in which base stations communicate with their mobile terminals. In this case, a transmitter T_1 wants to send information to its mobile R_1, while at the same time another base station T_2 (from another operator) wants to transmit information to its mobile R_2. These systems therefore share the same spectrum where the communication between the pairs (T_1, R_1) and (T_2, R_2) takes place simultaneously and on the same frequency band. This setup is known as the interference channel (IFC) ([1]- [3], [5] and [8] to mention a few).

There is a great deal of work on the IFC channel using game theory. In [3], the problem of power allocation in a frequency-selective multi-user interference channel is studied. An iterative Water-Filling (WF) algorithm is proposed to efficiently reach the Nash equilibrium. Moreover, it is found that under suitable conditions, the iterative WF algorithm for the two-user gaussian interference game converges to the unique Nash equilibrium from any starting point. In their scenario, the Nash equilibria lead to non-efficient and non pareto-optimal solutions. Similarly, in [1], the authors consider the problem of spectrum sharing on the IFC for flat-fading channels. The interference channel is viewed as a non-cooperative game and the Nash equilibrium is characterized under a set of sufficient conditions. In [6], the authors investigate the problem of simultaneous water-filling solution for the gaussian IFC under *weak* interference. Motivated by the pareto-inefficiency of the water-filling approach, the authors propose a distributed algorithm to transform a symmetric system from simultaneously waterfilled to a fair orthogonal signal space partitions.

In [10], the problem of two wireless networks operating on the same frequency band was considered. Pairs within a given network cooperate to schedule transmissions according to a random access protocol where each network chooses an access probability for its users. In [7], the authors consider the problem of coordinating two competing multiple-antenna wireless systems in the Multiple Input Single Output (MISO) IFC. It turns out that if the systems do not cooperate, then the corresponding equilibrium rates are bounded regardless of how much transmit power the base stations have available. Also, Nash bargaining solutions were found to be close to the sum-rate bound. On the other hand, in [12]- [14] the authors study the problem of maximizing mutual information subject to mask constraints and transmit power, for both simultaneous and asynchronous[1] cases. The existence of the Nash equilibrium is proven and sufficient conditions are given for the uniqueness. Finally, in [13], distributed iterative algorithms are proposed to reach the Nash equilibrium.

In most of these works, the existence of the Nash equilibrium is easily demonstrated, whereas the uniqueness is generally more complicated for which only sufficient conditions are given. Because of the very hard problem of the uniqueness of the Nash equilibrium points in the WF game, Nash bargaining (NBS) solutions were considered in [5]. However, NBS requires the knowledge of all channel state information which is not always possible in practice.

Within the same framework but under a different scenario, Stackelberg games [15] have been applied in the context of cognitive radios where the desirability of outcomes depends not only on their own actions but also on other cognitive radios. Stackelberg is furthermore based on a leader follower approach in which the leader plays his strategy before the follower and then enforces it. In [20], a game theoretic framework has been proposed in the context of fading multiple-access channel. A Stackelberg formulation is proposed in which the base station is the designated game leader with the purpose to

[1]Under this setup, some users are allowed to update their strategy more frequently than the others. And, they might even perform these updates using outdated information on the interference caused by others.

have a distributed allocation strategy approaching all corners of the capacity region. In [21], a two-level Stackelberg game is proposed for distributed relay selection and power control for multi-user cooperative networks. The objective is to jointly consider the benefits of source and relay nodes in which the source node is modeled as a buyer and the relay nodes as the sellers. Moreover, in [16], the authors investigate a similar power allocation problem but *solely* focus on channel realizations in which the Nash equilibrium of the game is unique. However, in this work the Stackelberg approach is mainly motivated by the non-uniqueness of the Nash equilibrium and unpredictability of the game.

In essence, the fundamental questions we address in this paper are the following:

- If both transmitters T_1 and T_2 simultaneously operate in a non-cooperative (i.e. *selfish*) manner, what are their power allocation strategies across their carriers? clearly, there is a conflict situation where a good strategy for the link (T_1, R_1) will generate interference for R_2 and vice-versa. Hence an equilibrium has to be found.
- Given any set of channel realizations, is it possible to predict the outcome of the game? if so, how to characterize the Nash equilibria regions? Is the Nash equilibrium *unique*?
- What is the outcome of the spectrum sharing game when operators do not play simultaneously, but hierarchically?
- How close is the distributed approach from the centralized (*sum-rate*) power allocation?.

The paper is organized as follows: The system model is introduced in Section II. In Section III, the spectrum sharing game is formulated using non-cooperative game theory. In Section IV, a special case with two transmitters and two carriers is investigated. Moreover, in Section V, we formulate the non-cooperative problem as a Stackelberg game to tackle the non-uniqueness of the Nash equilibrium. Finally, Section VI provides a comparison between the distributed (selfish) and centralized approach. Finally, we conclude this work in Section VII.

II. System Model

We suppose that K transmitters[2] share a frequency band composed of N carriers where each transmitter transmits in any combination of channels and at any time. On each carrier $n = 1..N$, transmitter $i = 1..K$ sends the information $x_i^n = \sqrt{p_i^n} s_i^n$, where s_i^n represents the transmitted data and p_i^n denotes the corresponding transmitted power of user i on carrier n. The received signal at the receiver i in carrier n can be expressed as:

$$r_i^n = \sum_{j=1}^{K} h_{ji}^n x_j^n + w_i^n, \quad i = 1,..,K \quad n = 1,..,N \quad (1)$$

where h_{ji}^n is the fading channel gain on the n^{th} carrier between the pair $(T_i$,$R_j)$. In addition, the noise process w_i^n

[2]The terms transmitter and operator are intercheanbly used in this paper

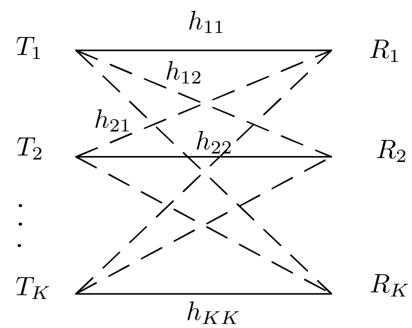

Fig. 1. K-Transmitters N-Carriers interference channel under study.

is characterized by its received noise power on each carrier n, by σ_n^2.

For user k, the transmit power is subject to its power constraint:

$$\sum_{n=1}^{N} p_i^n \leq \bar{P}_i, \qquad i = 1,..,K \quad (2)$$

At the receiver i, the signal to interference plus noise ratio (SINR) on carrier n is given by:

$$SINR_i^n = \frac{p_i^n |h_{ii}^n|^2}{\sigma_n^2 + \sum_{j=1,j\neq i}^{K} p_j^n |h_{ji}^n|^2} \quad (3)$$

for each user, given that all other users use Gaussian codebooks, the codebook that maximizes mutual information is also Gaussian [26]. Furthermore, (assuming static links) the maximum achievable rate at receiver i is given by:

$$R_i = \sum_{n=1}^{N} \log_2(1 + SINR_i^n) \quad (4)$$

III. Non-Cooperative Game

In this section, we model the spectrum sharing problem from a non-cooperative standpoint [22]. Figure 1 illustrates the spectrum sharing scenario under study.

A. Game Formulation

The non-cooperative spectrum sharing game is defined as: $\Gamma^{NCG} \triangleq [\mathcal{K}, \{\mathcal{P}_i\}_{i\in\mathcal{K}}, \{\mathcal{U}_i\}_{i\in\mathcal{K}}]$. The players (from the set $\mathcal{K} \triangleq \{1, 2, .., K\}$) are defined as the different links with a strategy $p_i^n \in \mathcal{P}_i$ and the payoffs are the achievable rates on each link $u_i(p_i^n, p_{-i}^n) = R_i(p_i^n, p_{-i}^n) \in \mathcal{U}_i$, for $i = 1, ..K$ and $n = 1, ..N$. The notation *"-i"* denotes the player other than i. Each player competes against the others by choosing his transmit power (i.e., strategy) to maximize his own utility subject to some power constraints $\bar{P}_i$. In this work, we assume full channel state information.

Since the transmitters do not cooperate, the only reasonable outcome of the spectrum conflict is an operating point which constitutes a Nash Equilibrium (N.E) [24]. This is a point where none of the players can improve their utilities by unilaterally changing their strategies. One should note that a N.E is not an optimal or even desirable outcome. However, it is an insightful point where one is likely to end up operating if both players are not willing to cooperate.

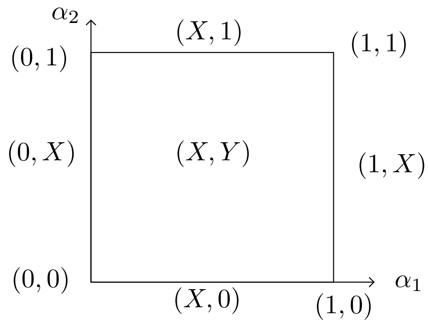

Fig. 2. Illustration of the Nash Equilibria Space where (α_1, α_2) denotes the power allocation strategy for both operators 1 and 2, on the first carrier.

In a non-cooperative approach, user 1 selfishly maximizes his utility function subject to the power constraint $\bar{P}_1$:

$$\begin{aligned} \max_{p_1^n} R_1 = \quad & \max_{p_1^n} \sum_{n=1}^N \log_2\left(1 + \frac{|h_{11}^n|^2 p_1^n}{\sigma_n^2 + \sum_{j=1, j\neq 1}^K |h_{j1}^n|^2 p_j^n}\right) \\ s.t. \quad & \sum_{n=1}^N p_1^n \leq \bar{P}_1 \\ & p_1^n \geq 0 \end{aligned} \tag{5}$$

Furthermore, the channel realization set $\underline{h}$ is defined as:

$$\underline{h} = \{h_{ij}^n : \quad i,j = 1,..,K, \quad n = 1,..,N\} \tag{6}$$

Likewise for user 2, the rate maximization problem is given by:

$$\begin{aligned} \max_{p_2^n} R_2 = \quad & \max_{p_2^n} \sum_{n=1}^N \log_2\left(1 + \frac{|h_{22}^n|^2 p_2^n}{\sigma_n^2 + \sum_{j=1, j\neq 2}^K |h_{j2}^n|^2 p_j^n}\right) \\ s.t. \quad & \sum_{n=1}^N p_2^n \leq \bar{P}_2 \\ & p_2^n \geq 0 \end{aligned} \tag{7}$$

The solutions to Equations (5)-(7) are given by the water-filling power allocation solutions:

$$p_i^n = \left(\frac{1}{\mu_i} - \frac{\sigma_n^2 + \sum_i |h_{-ii}^n|^2 p_{-i}^n}{|h_{ii}^n|^2}\right)^+ \quad i = 1,..,K \quad n = 1,..,N \tag{8}$$

where $(x)^+ = \max\{x, 0\}$ and $\mu_i > 0$ is the Lagrangian multiplier chosen to satisfy the power constraint: $\sum_{n=1}^N p_i^n = \bar{P}_i$. Note that the equality follows from the concavity of the objective function in p_i. Finally, (8) represents the best response of player p_i given p_{-i}.

IV. Special Case of Two Transmitters and Two Carriers

In order to gain insight into the properties of the Nash equilibria for our game, we focus on a system with two operators and two carriers (i.e., $K = N = 2$).

A. Notations

- For ease of notation and readability that will prove helpful in the sequel, we introduce the following notations: $g_{ij}^n = \frac{\bar{P}_i |h_{ij}^n|^2}{\sigma_n^2}$, $c_1 = \frac{g_{11}^1}{g_{11}^2}$ and $c_2 = \frac{g_{22}^1}{g_{22}^2}$.

- The pair (α_1,α_2) means that user 1 transmits with power $(p_1^1, p_1^2) = (\alpha_1 \bar{P}_1, (1-\alpha_1)\bar{P}_1)$ on carrier 1 and 2 while user 2 transmits with power $(p_2^1, p_2^2) = (\alpha_2 \bar{P}_2, (1-\alpha_2)\bar{P}_2)$ on carrier 1 and 2, respectively.

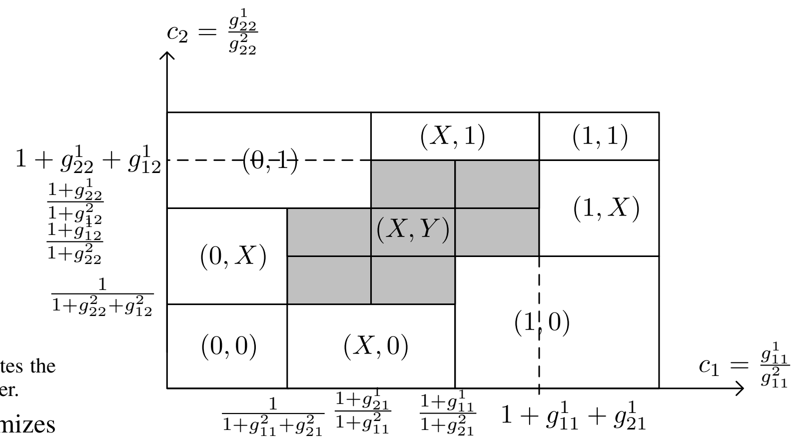

Fig. 3. Characterization of the Nash equilibria regions given a set of channel realizations $\underline{h}$.

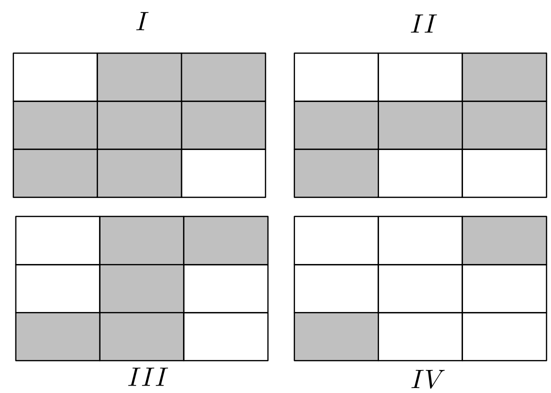

Fig. 4. All four cases are depicted: I) when $\frac{1+g_{11}^1}{1+g_{21}^2} > \frac{1+g_{21}^1}{1+g_{11}^2}$ and $\frac{1+g_{22}^1}{1+g_{12}^2} > \frac{1+g_{12}^1}{1+g_{22}^2}$, II) when $\frac{1+g_{11}^1}{1+g_{21}^2} < \frac{1+g_{21}^1}{1+g_{11}^2}$ and $\frac{1+g_{22}^1}{1+g_{12}^2} > \frac{1+g_{12}^1}{1+g_{22}^2}$, III) when $\frac{1+g_{11}^1}{1+g_{21}^2} > \frac{1+g_{21}^1}{1+g_{11}^2}$ and $\frac{1+g_{22}^1}{1+g_{12}^2} < \frac{1+g_{12}^1}{1+g_{22}^2}$ and finally, IV) $\frac{1+g_{11}^1}{1+g_{21}^2} < \frac{1+g_{21}^1}{1+g_{11}^2}$ and $\frac{1+g_{22}^1}{1+g_{12}^2} < \frac{1+g_{12}^1}{1+g_{22}^2}$.

Figure 2 depicts the space of the 9 Nash equilibria of the game obtained upon solving Equation (8), the details of which are given in appendix A. Given a set of channel realizations $\underline{h}$, the game converges to different equilibrium points. Figure 3 illustrates one possible representation of the Nash equilibria space. Depending on the quantities $\frac{1+g_{21}^1}{1+g_{11}^2}$, $\frac{1+g_{11}^1}{1+g_{21}^2}$ and $\frac{1+g_{22}^1}{1+g_{12}^2}$, $\frac{1+g_{12}^1}{1+g_{22}^2}$ (see appendix A) four different representation of the regions are possible. These regions are depicted in Figure 4 whose purpose is to reflect the 8-dimensional problem related to the channel realization set $\underline{h}$.

It turns out that given certain channel realizations, the Nash equilibrium is *unique* (white rectangle areas) while some of the grayish rectangle regions exhibit at least one Nash equilibria.

B. Existence of the Nash Equilibria

The existence of the Nash equilibria is proven using the theorem in [25] within the context of non-cooperative concave games. Hence, the game defined in (4) admits at least one Nash equilibrium.

C. Uniqueness of the Nash Equilibria

As previously mentioned, proving the uniqueness of the N.E is in general difficult [5]. In [3], the authors give sufficient[3] conditions for the uniqueness of the N.E. However, the authors do not precisely state which N.E are obtained for any given channel realization set $\underline{h}$. Therefore building on these results, a full characterization of the Nash equilibria region for the 2 users 2 carriers case is highlighted herein. In addition, the proof of the uniqueness when both operators transmit in both carriers is given in appendix B. Finally, it is worth mentioning that the sufficient conditions given for the flat-fading case studied in [1] are depicted in Figure 3 for the low-interferece regime (X,Y).

On the other hand, we note that when one of the cross-gain $|h_{-i,i}|^2 = 0$, the IFC becomes a Z-channel [19] where the N.E. exists and is *unique* (the characterization of the Nash equilibria region for the Z-channel follows the same lines as the IFC).

V. Stackelberg Game

In the previous section, the users were assumed to be non-cooperative hence operating at the Nash Equilibrium was the best response of a user in a selfish context, when the game is played simultaneously. It was also shown that in some regions and given a set of channel realizations, the game is predictable with a unique Nash equilibrium. However, in other regions and given a set of channel realizations, *non-unique* Nash equilibria exist. In this case, the spectrum sharing game is *no longer* predictable.

Motivated by this result, a *Stackelberg* game $\Gamma^{SG} \triangleq [\mathcal{K}, \{\mathcal{P}_i\}_{i\in\mathcal{K}}, \{\mathcal{U}_i\}_{i\in\mathcal{K}}]$ is proposed to model the spectrum sharing problem where one of the two users is chosen to be the leader. The *Stackelberg* Equilibrium (SE) [15] is the best response where a hierarchy of actions exists between players. Backward induction is applied assuming that players can reliably forecast the behavior of other players and that they believe that the other player can do the same. For this reason, the key point in this setup is the capability of the follower of sensing the environment and therefore the power level of user 1 (the leader). Note that in the Stackelberg approach all channels should be known[4].

A. Problem Formulation

Without loss of generality, we assume that T_1 is the leader and T_2 is the follower[5].

Definition 1: (**Stackelberg Equilibrium**) [22] A strategy profile (p_1^{SE},p_2^{SE}) is called a Stackelberg Equilibrium if p_1^{SE} maximizes the utility of the leader (user 1) and p_2^{SE} is the best response of user 2 to user 1.

The Stackelberg spectrum sharing game can be formulated as follows. First, in the high-level problem (9), operator 1 maximizes his own utility function. Then, in the low-level problem (10), operator 2 (follower) maximizes his own utility taking into account the optimal power allocation of operator 1 (p_1^{SE}). By denoting (p_1^{SE}, p_2^{SE}) as the Stackelberg Equilibrium, the rate optimization problem for operator 1 (leader) writes as:

$$\max_{p_1^n} \sum_{n=1}^{N} \log_2\left(1 + \frac{|h_{11}^n|^2 p_1^n}{\sigma_n^2 + |h_{21}^n|^2 p_2^n}\right) \tag{9}$$

$$\sum_{n=1}^{N} p_1^n \leq \bar{P}_1$$

$$p_1^n \geq 0$$

The rate optimization problem for operator 2 (follower) writes as:

$$\max_{p_2^n} \sum_{n=1}^{N} \log_2\left(1 + \frac{|h_{22}^n|^2 p_2^n}{\sigma_n^2 + |h_{12}^n|^2 \left(p_1^n\right)^{SE}}\right) \tag{10}$$

$$\sum_{n=1}^{N} p_2^n \leq \bar{P}_2$$

$$p_2^n \geq 0$$

where $p_2^{SE} = BR_2(p_1^{SE})$.

Using backward induction and given the best response of operator 2 (the follower), (10) can be rewritten as:

$$\max_{p_1^n} \sum_{n=1}^{N} \log_2\left(1 + \frac{|h_{11}^n|^2 p_1^n}{\sigma_n^2 + |h_{21}^n|^2 \left(\frac{1}{\mu_2} - \frac{\sigma_n^2 + |h_{12}^n|^2 p_1^n)}{|h_{22}^n|^2}\right)^+}\right) \tag{11}$$

$$\sum_{n=1}^{N} p_1^n \leq \bar{P}_1$$

$$p_1^n \geq 0$$

The Stackelberg game boils down to solving (11). To this end, several cases are considered. In our spectrum sharing problem ($K = N = 2$), the power strategies of operator 2 take 3 values. In the first case, operator 2 transmits with maximum power $\bar{P}_2$ in carrier 1 such that $(p_2^1 = \bar{P}_2, p_2^2 = 0)$. In the second case, operator 2 transmits with $\bar{P}_2$ in carrier 2 such that $(p_2^1 = 0, p_2^2 = \bar{P}_2)$ and finally in the third case, operator 2 transmits with $(p_2^1 = x, p_2^2 = \bar{P}_2 - x)$, $0 < x < \bar{P}_2$. Therefore, the leader maximizes his utility function given the best response of the follower. In the following, the *three* cases are investigated:

1) **Operator 2 transmits only in carrier 2** $(\mathbf{p_2^1 = 0, p_2^2 = 1})$

[3] The N.E is unique if $\rho(S(k)) < 1$, for all k $\in \{1, .., N\}$ where $S(k)$ is given in [14] and ρ is the spectral radius.

[4] This assumption can be considered strong. But it is in the advantage of operators to sense the environment and think in a long-term prospect rather than being myopic.

[5] The other case is similar. One can think of a TDMA approach or round-robin where users leadership is alternated.

Under this setup, $p_2^2 > 0 \Rightarrow p_1^1 \geq \beta_1$ where:

$$\beta_1 = \frac{\frac{\sigma_2^2}{|h_{22}^2|^2} + \frac{|h_{12}^2|^2}{|h_{22}^2|^2} - \frac{\sigma_1^2}{|h_{22}^1|^2} + 1}{\frac{|h_{12}^1|^2}{|h_{22}^1|^2} + \frac{|h_{12}^2|^2}{|h_{22}^2|^2}} \tag{12}$$

Furthermore, the maximization problem for the leader is written as:

$$\max_{p_1^1} \quad \log_2\left(1 + \frac{|h_{11}^1|^2 p_1^1}{\sigma_1^2}\right) + \log_2\left(1 + \frac{|h_{11}^2|^2(\bar{P}_1 - p_1^1)}{\sigma_2^2 + |h_{21}^2|^2}\right)$$
$$\max(\beta_1, 0) \leq p_1^1 \leq \bar{P}_1 \tag{13}$$

the Karush-Kuhn-Tucker (KKT) [23] conditions are given by:

$$\begin{cases} \lambda_1^*((p_1^1)^* - \max(\beta_1, 0)) = 0, \quad \lambda_1^* \geq 0 \\ \lambda_2^*((p_1^1)^* - \bar{P}_1) = 0, \quad \lambda_2^* \geq 0 \\ \frac{\partial \mathcal{L}_1}{\partial p_1^1} = 0, \quad (p_1^1)^* \leq \bar{P}_1 \\ (p_1^1)^* \geq \max(\beta_1, 0) \end{cases}$$

where λ_1^*, λ_2^* are the Lagrangian multipliers associated with the constraints given above.

$$\mathcal{L}_1 = \log_2\left(1 + \frac{|h_{11}^1|^2 p_1^1}{\sigma_1^2}\right) + \log_2\left(1 + \frac{|h_{11}^2|^2(\bar{P}_1 - p_1^1)}{\sigma_2^2 + |h_{21}^2|^2}\right)$$
$$-\lambda_1^*(p_1^1 - \max(\beta_1, 0)) + \lambda_2^*(p_1^1 - \bar{P}_1) \tag{14}$$

$$\frac{\partial \mathcal{L}_1}{\partial p_1^1} = 0 \Rightarrow \frac{|h_{11}^1|^2}{\sigma_1^2 + |h_{11}^1|^2 p_1^1} -$$
$$\frac{|h_{11}^2|^2}{\sigma_2^2 + |h_{21}^2|^2 + |h_{11}^2|^2(\bar{P}_1 - p_1^1)} = \lambda_1^* - \lambda_2^* \tag{15}$$

Assume that $p_1^1 = \max(\beta_1, 0)$. Then, $\lambda_1^* \geq 0, \lambda_2^* = 0$ and:

$$\frac{|h_{11}^1|^2}{\sigma_1^2 + |h_{11}^1|^2 \max(\beta_1, 0)} -$$
$$\frac{|h_{11}^2|^2}{\sigma_2^2 + |h_{21}^2|^2 + |h_{11}^2|^2(\bar{P}_1 - \max(\beta_1, 0))} \geq 0 \tag{16}$$

Now assuming that $p_1^1 = \bar{P}_1$, then $\lambda_1^* = 0, \lambda_2^* \geq 0$ and:

$$\frac{|h_{11}^1|^2}{\sigma_1^2 + |h_{11}^1|^2 \bar{P}_1} - \frac{|h_{11}^2|^2}{\sigma_2^2 + |h_{21}^2|^2} \leq 0 \tag{17}$$

Finally, assuming that $\max(\beta_1, 0) < p_1^1 < \bar{P}_1$, then $\lambda_1^* = \lambda_2^* = 0$ and:

$$p_1^1 = \frac{\sigma_1^2 |h_{11}^1|^2 - \sigma_2^2 |h_{11}^2|^2 + |h_{11}^1|^2(|h_{21}^2|^2 + |h_{11}^2|^2 \bar{P}_1)}{2|h_{11}^1|^2 |h_{11}^2|^2} \tag{18}$$

2) **Operator** 2 **transmits only in carrier** 1 ($\mathbf{p_2^1 = 1, p_2^2 = 0}$)
Under this setup, $p_2^1 > 0 \Rightarrow p_1^1 \leq \beta_2$ where

$$\beta_2 = \frac{\frac{\sigma^2}{|h_{22}^2|^2} + \frac{|h_{12}^2|^2}{|h_{22}^2|^2} - \frac{\sigma^2}{|h_{22}^1|^2} - 1}{\frac{|h_{12}^1|^2}{|h_{22}^1|^2} + \frac{|h_{12}^2|^2}{|h_{22}^2|^2}} \tag{19}$$

Furthermore, the maximization problem for the leader writes as:

$$\max_{p_1^1} \log_2\left(1 + \frac{|h_{11}^1|^2 p_1^1}{\sigma_1^2 + |h_{21}^1|^2}\right) + \log_2\left(1 + \frac{|h_{11}^2|^2(\bar{P}_1 - p_1^1)}{\sigma_2^2}\right)$$
$$0 \leq p_1^1 \leq \min(\beta_2, \bar{P}_1) \tag{20}$$

Likewise, to derive the KKT conditions, form the Lagrangian denoted as $\mathcal{L}_2$.

$$\mathcal{L}_2 = \log_2\left(1 + \frac{|h_{11}^1|^2 p_1^1}{\sigma_1^2 + |h_{21}^1|^2}\right) + \log_2\left(1 + \frac{|h_{11}^2|^2(\bar{P}_1 - p_1^1)}{\sigma_2^2}\right)$$
$$-\lambda_1^* p_1^1 + \lambda_2^*(p_1^1 - \min(\beta_2, \bar{P}_1)) \tag{21}$$

the KKT conditions are:

$$\begin{cases} \lambda_2^*((p_1^1)^* - \min(\beta_2, \bar{P}_1)) = 0, \quad \lambda_2^* \geq 0 \\ \lambda_1^*((p_1^1)^*) = 0, \quad \lambda_1^* \geq 0 \\ \frac{\partial \mathcal{L}_2}{\partial p_1^1} = 0, \\ (p_1^1)^* \geq 0, \\ (p_1^1)^* \leq \min(\beta_2, \bar{P}_1) \end{cases}$$

where λ_1^*, λ_2^* are the Lagrangian multipliers associated with the constraints.

$$\frac{\partial \mathcal{L}_2}{\partial p_1^1} = 0 \Rightarrow \frac{|h_{11}^1|^2}{\sigma_1^2 + |h_{21}^1|^2 + |h_{11}^1|^2 p_1^1} -$$
$$\frac{|h_{11}^2|^2}{\sigma_2^2 + |h_{11}^2|^2(\bar{P}_1 - p_1^1)} = \lambda_1^* - \lambda_2^* \tag{22}$$

Assume that $p_1^1 = 0$, $\lambda_1^* \geq 0$, then $\lambda_2^* = 0$ and furthermore:

$$\frac{|h_{11}^1|^2}{\sigma_1^2 + |h_{21}^1|^2} - \frac{|h_{11}^2|^2}{\sigma_2^2 + |h_{11}^2|^2 \bar{P}_1} \geq 0 \tag{23}$$

Assuming that $p_1^1 = \min(\beta_2, \bar{P}_1)$, $\lambda_2 \geq 0$, then $\lambda_1 = 0$ and furthermore:

$$\frac{|h_{11}^1|^2}{\sigma_1^2 + |h_{21}^1|^2 + |h_{11}^1|^2 \min(\beta_2, \bar{P}_1)} -$$
$$\frac{|h_{11}^2|^2}{\sigma_2^2 + |h_{11}^2|^2(\bar{P}_1 - \min(\beta_2, \bar{P}_1))} = \lambda_1^* - \lambda_2^* \tag{24}$$

Finally, assume that $0 < p_1^1 < \min(\beta_2, \bar{P}_1)$, then $\lambda_1^* = \lambda_2^* = 0$ and:

$$p_1^1 = \frac{|h_{11}^1|^2 \sigma_2^2 - |h_{11}^2|^2 \sigma_1^2 + |h_{11}^2|^2(|h_{11}^1|^2 \bar{P}_1 - |h_{21}^1|^2)}{2|h_{11}^1|^2 |h_{11}^2|^2} \tag{25}$$

3) **Operator** 2 **transmits in both carriers** ($\mathbf{p_2^1 = x, p_2^2 = 1 - x}$)

$$\max_{p_1^1} \log_2\left(1 + \frac{|h_{11}^1|^2 p_1^1}{\sigma_1^2 + |h_{21}^1|^2 x}\right) + \log_2\left(1 + \frac{|h_{11}^2|^2 p_1^2}{\sigma_2^2 + |h_{21}^2|^2(1 - x)}\right)$$
$$p_1^1 + p_1^2 \leq \bar{P}_1,$$
$$p_1^1, p_1^2 \geq 0,$$
$$\beta_2 < p_1^1 < \beta_1 \tag{26}$$

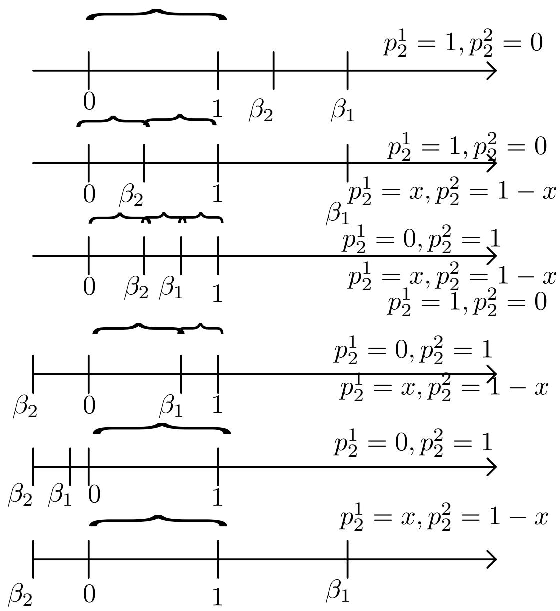

Fig. 5. Power allocation strategies for the Stackelberg game in which 6 cases exist depending on the variables β_1 and β_2 ($\beta_2 < \beta_1$). The X-axis depicts the strategy space for the leader transmitting with power p_1^1.

Since $p_2^1 = x = \frac{1}{\mu_2} - \frac{\sigma^2 + |h_{12}^1|^2 P_1^1}{|h_{22}^1|^2} > 0$ depends on p_1^1, the objective function (26) of user 1 is non-convex in p_1^1 (the KKT conditions can be written in the same way as done for the previous cases and the problem is solved numerically). Figure 5 depicts all of the 6 different cases depending on the values of β_1 and β_2 ($\beta_2 < \beta_1$).

As can be seen, in the first case ($\beta_2 > 1$, $\beta_1 > 1$) the leader has to perform one maximization over the interval $[0, 1]$. In the *second* case ($0 < \beta_2 < 1$, $\beta_1 > 1$), the leader has to perform 2 maximizations ($[0, \beta_2]$ and $[\beta_2, 1]$) and pick the power allocation that maximizes his payoff. Similarly, the leader has 3 maximizations to perform in the *third* case ($[0, \beta_2]$, $[\beta_2, \beta_1]$ and $[\beta_1, 1]$) where $0 < \beta_1 < 1$, $0 < \beta_2 < 1$ and likewise for the remaining cases. In essence, in all these cases, the leader (user 1) forces the follower to adopt a strategy that maximizes the leader's payoff. In this way, the Stackelberg equilibrium is *unique*, solving thereby the problem of non-uniqueness encountered in the non-cooperative approach of Section IV. Additionally, one should note that there exist Stackelberg solutions that are *non-Nash* equilibria of the non-cooperative game.

VI. Numerical Evaluation

In this section, numerical results are presented to validate the theoretical claims. Figure 4 depicts the average achievable rate of both operators for the Stackelberg approach. In the simulations, we let the individual power constraint $\bar{P}_1 = \bar{P}_2 = \bar{P} = 1$, $SNR = \frac{\bar{P}}{\sigma^2}$ and channel fading realizations are independent and identically distributed (i.i.d) Rayleigh distributed.

It is important to quantify the performance loss from the optimal solution provided by the centralized strategy. To this end, we compare the Stackelberg rates with the rates obtained by sum-rate maximization (which are Pareto-optimal):

$$\begin{aligned} \max_{p_1^n, p_2^n} \quad & \sum_{i=1}^{K} \sum_{n=1}^{N} \log_2 \left(1 + \frac{|h_{ii}^n|^2 p_i^n}{\sigma_n^2 + |h_{-ii}^n|^2 p_{-i}^n}\right) \\ & \sum_{n=1}^{N} p_1^n \leq \bar{P}_1, \quad \sum_{n=1}^{N} p_2^n \leq \bar{P}_2, \\ & p_1^n \geq 0, \quad p_2^n \geq 0, \quad n = 1, .., N \end{aligned} \tag{27}$$

The objective function is non-convex in the power variables p_1^n and p_2^n. To solve (27) the maximization problem is transformed into a convex optimization problem using Geometric Programming [23].

Additionally, Figure 6 depicts the best and worst N.E where the best N.E refers to the equilibrium maximizing the sum-rate of both operators whereas the worst N.E case minimizes it. It is also worth noting that the worst Nash equilibrium acts like a lower-bound for the Nash equilibrium. Furthermore, the Stackelberg approach is closer to the centralized approach as compared to the selfish case. This is due to the fact that in the Stackelberg approach, operators take into account other operators' strategies whereas in the selfish case, operators behave carelessly by using water-filling.

Figure 7 shows the achievable rate region for both operators in which the Nash and Stackelberg equilibria are illustrated. Since operator 1 is the leader, his rate is higher with the Stackelberg approach. Also, interestingly, the rate of operator 2 is also better off with the Stackelberg approach. As a result, both operators have strong incentives in adopting the hierarchical (Stackelberg) approach

VII. Conclusion

In this paper, we have studied the problem of spectrum sharing between operators operating in the same frequency band. First, a *one-shot* game was studied where the players play simultaneously, operating at the Nash equilibrium point. It was found that the Nash equilibria regions exhibit different behaviors according to the set of channel realizations. Some regions have unique N.E whereas others have many. To solve the non-unique characteristic of the game, a Stackelberg game is proposed where one of the operators (leader) is firstly deployed in the network while the other one (follower) is deployed next.

In our future work, we will focus on the same spectrum sharing problem using repeated games in an effort to approach the Pareto-optimal solution.

VIII. Acknowledgment

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Appendix A

We derive the set of 9 inequalities for the Nash equilibria when $K = 2$ users are transmitting over $N = 2$ carriers, for the non-cooperative game Γ^{NC}.

$(\alpha_1, \alpha_2) = (0, 0)$ is a Nash Equilibrium $\Leftrightarrow$

$$\begin{cases} c_1 \leq \frac{1}{1 + g_{11}^2 + g_{21}^2} \\ c_2 \leq \frac{1}{1 + g_{22}^2 + g_{12}^2} \end{cases}$$

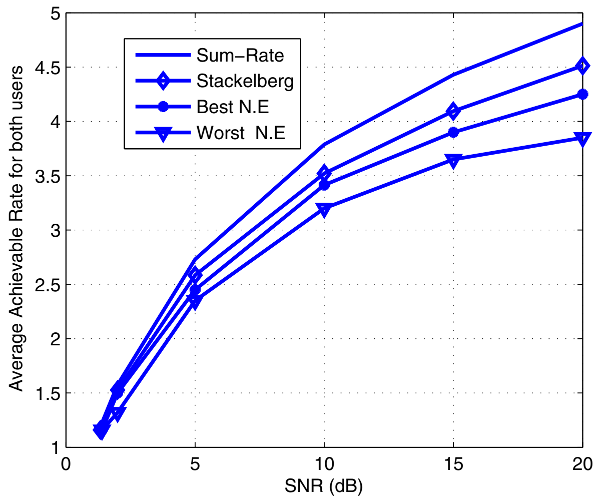

Fig. 6. Average achievable rate for both users versus the signal-to-noise ratio for the centralized and Stackelberg approach. Moreover, the best and worst Nash equilibria for the non-cooperative game are illustrated.

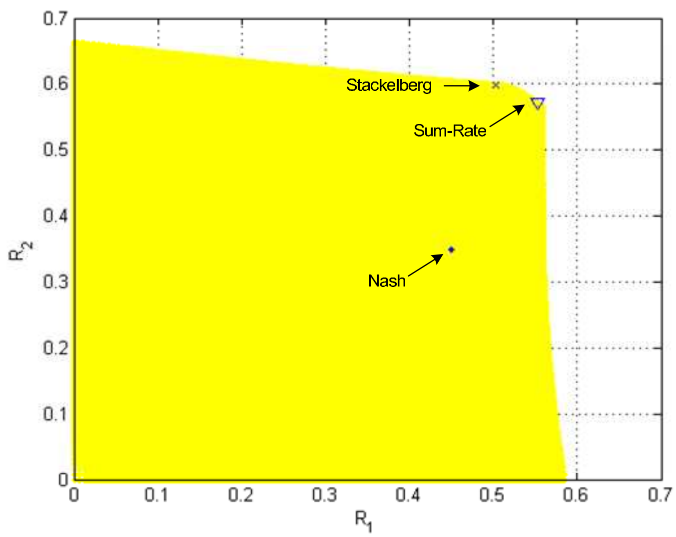

Fig. 7. Achievable rate region for the inter-operator spectrum sharing game. Both operators achieve better payoffs when adopting the hierarchical (Stackelbeg) approach.

$(\alpha_1, \alpha_2) = (1, 0)$ is a Nash Equilibrium $\Leftrightarrow$

$$\begin{cases} c_1 \geq \frac{1+g_{11}^1}{1+g_{21}^2} \\ c_2 \leq \frac{1+g_{12}^1}{1+g_{22}^2} \end{cases}$$

$(\alpha_1, \alpha_2) = (0, 1)$ is a Nash Equilibrium $\Leftrightarrow$

$$\begin{cases} c_1 \leq \frac{1+g_{21}^1}{1+g_{11}^2} \\ c_2 \geq \frac{1+g_{22}^1}{1+g_{12}^2} \end{cases}$$

$(\alpha_1, \alpha_2) = (1, 1)$ is a Nash Equilibrium $\Leftrightarrow$

$$\begin{cases} c_1 \geq 1 + g_{11}^1 + g_{21}^1 \\ c_2 \geq 1 + g_{22}^1 + g_{12}^1 \end{cases}$$

$(\alpha_1, \alpha_2) = (x, 1)$ is a Nash Equilibrium $\Leftrightarrow$

$$\begin{cases} 2g_{22}^1 g_{11}^1 \leq c_2 g_{11}^1(1 + g_{12}^2) - 2g_{11}^1 - (c_2 g_{12}^2 + g_{12}^1)\times \\ (c_1 - 1 + c_1 g_{11}^2 - g_{21}^1) \\ \frac{1+g_{21}^1}{1+g_{11}^2} \leq c_1 \leq 1 + g_{11}^1 + g_{21}^1 \end{cases}$$

where $x = \frac{1}{2} - \frac{1}{2g_{11}^1} - \frac{g_{21}^1}{2g_{11}^1} + \frac{1}{2g_{11}^2}$

$(\alpha_1, \alpha_2) = (1, x)$ is a Nash Equilibrium $\Leftrightarrow$

$$\begin{cases} 2g_{11}^1 g_{22}^1 \leq c_1 g_{22}^1(1 + g_{21}^2) - 2g_{22}^1 - (c_1 g_{21}^2 + g_{21}^1)\times \\ (c_2 - 1 + c_2 g_{22}^2 - g_{12}^1) \\ \frac{1+g_{12}^1}{1+g_{22}^2} \leq c_2 \leq 1 + g_{22}^1 + g_{12}^1 \end{cases}$$

where $x = \frac{1}{2} - \frac{1}{2g_{22}^1} - \frac{g_{12}^1}{2g_{22}^1} + \frac{1}{2g_{22}^2}$

$(\alpha_1, \alpha_2) = (0, x)$ is a Nash Equilibrium $\Leftrightarrow$

$$\begin{cases} \frac{c_2(1+g_{12}^2)+g_{22}^1-1}{2g_{22}^1} \geq \frac{c_1(1+g_{21}^2+g_{11}^2)-1}{g_{21}^1+c_1 g_{21}^2} \\ \frac{1}{1+g_{22}^2+g_{12}^2} \leq c_2 \leq \frac{1+g_{22}^1}{1+g_{12}^2} \end{cases}$$

where $x = \frac{1}{2} + \frac{g_{12}^2}{2g_{22}^2} - \frac{1}{2g_{22}^1} + \frac{1}{2g_{22}^2}$

$(\alpha_1, \alpha_2) = (x, 0)$ is a Nash Equilibrium $\Leftrightarrow$

$$\begin{cases} \frac{c_1(1+g_{21}^2)+g_{11}^1-1}{2g_{11}^1} \geq \frac{c_2(1+g_{12}^2+g_{22}^2)-1}{g_{12}^1+c_2 g_{12}^2} \\ \frac{1}{1+g_{11}^2+g_{21}^2} \leq c_1 \leq \frac{1+g_{11}^1}{1+g_{21}^2} \end{cases}$$

where $x = \frac{1}{2} + \frac{g_{21}^2}{2g_{11}^2} - \frac{1}{2g_{11}^1} + \frac{1}{2g_{11}^2}$

$(\alpha_1, \alpha_2) = (x, y)$ is a N.E $\Leftrightarrow \begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases}$

$$x = \frac{(2g_{21}^1(c1(1+g_{12}^2+g_{21}^2)-1)-(c_1 g_{21}^2+g_{21}^1)(c_2(1+g_{12}^2)+g_{22}^1-1))}{(4g_{11}^1 g_{22}^1-(c_2 g_{12}^2+g_{12}^1)(c_1 g_{21}^2+g_{21}^1))}$$

$$y = \frac{(1/(2g_{22}^2)-1/(2g_{22}^1)+((1-x)g_{12}^2)}{(2g_{22}^1)-(g_{12}^1 x)/2g_{22}^1+\frac{1}{2})}$$

APPENDIX B

In this setup, the utility functions become:

$$\begin{cases} R_1(\alpha_1, \alpha_2) = \log_2(1 + \frac{|h_{11}^1|^2\alpha_1}{\sigma_1^2+|h_{21}^1|^2\alpha_2}) + \log_2(1 + \frac{|h_{11}^2|^2(1-\alpha_1)}{\sigma_2^2+|h_{21}^2|^2(1-\alpha_2)}) \\ R_2(\alpha_1, \alpha_2) = \log_2(1 + \frac{|h_{22}^1|^2\alpha_2}{\sigma_1^2+|h_{12}^1|^2\alpha_1}) + \log_2(1 + \frac{|h_{22}^2|^2(1-\alpha_2)}{\sigma_2^2+|h_{12}^2|^2(1-\alpha_1)}) \end{cases}$$

We will give now sufficient conditions that guarantee the uniqueness of the N.E. By analyzing the first order derivatives of the payoff functions, we can find explicit relations for the best response functions (BR):

$$\begin{cases} BR_1(\alpha_2) = \frac{-\left[|h_{11}^2|^2|h_{21}^1|^2+|h_{11}^1|^2|h_{21}^2|^2\right]\alpha_2-|h_{11}^2|^2+|h_{11}^1|^2(1+|h_{21}^2|^2+|h_{11}^2|^2)}{2|h_{11}^1|^2|h_{11}^2|^2} \\ BR_2(\alpha_1) = \frac{-\left[|h_{22}^2|^2|h_{12}^1|^2+|h_{22}^1|^2|h_{12}^2|^2\right]\alpha_1-|h_{22}^2|^2+|h_{22}^1|^2(1+|h_{12}^2|^2+|h_{22}^2|^2)}{2|h_{22}^1|^2|h_{22}^2|^2} \end{cases}$$

We observe that the functions $BR_i(\alpha_{-i})$ are linear w.r.t. α_{-i}. Thus, the intersection of the BR functions is either a unique point or an infinity of points. Therefore, the sufficient conditions that ensure the uniqueness of the N.E are the following:

$$\begin{cases} \frac{|h_{11}^2|^2|h_{21}^1|^2+|h_{11}^1|^2|h_{21}^2|^2}{2|h_{11}^1|^2|h_{11}^2|^2} \neq \frac{2|h_{22}^1|^2|h_{22}^2|^2}{|h_{22}^2|^2|h_{12}^1|^2+|h_{22}^1|^2|h_{12}^2|^2} \\ \frac{-|h_{11}^2|^2+|h_{11}^1|^2(1+|h_{21}^2|^2+|h_{11}^2|^2)}{2|h_{11}^1|^2|h_{11}^2|^2} \neq \frac{-|h_{22}^2|^2+|h_{22}^1|^2(1+|h_{12}^2|^2+|h_{22}^2|^2)}{|h_{22}^2|^2|h_{12}^1|^2+|h_{22}^1|^2|h_{12}^2|^2} \end{cases}$$

If these conditions are met, the unique point at the intersection of the BRs describes the Nash equilibrium. This is illustrated in Figure 8.

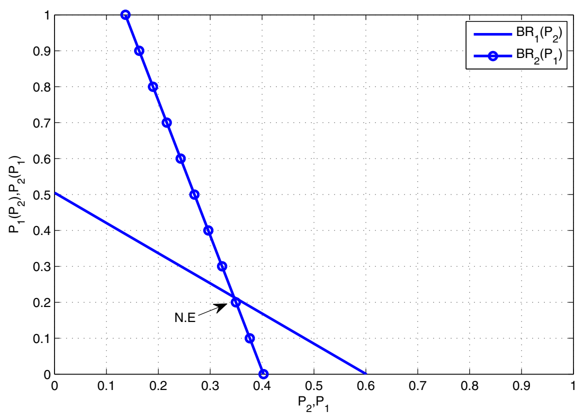

Fig. 8. Best response functions illustrating the unique Nash equilibrium point where both operators transmit in both carriers.

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