Anti-windup control design for exponentially unstable LTI systems with actuator saturation: the non-strictly proper case

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***Abstract*— We consider the control problem of the design of an anti-windup compensator for exponentially unstable linear systems subject to input saturation. We revisit the results in [1] and we generalize the LMI conditions for an anti-windup design that explicitly takes into account the presence of a direct feedthrough term in the plant's dynamical model, from the control input to the measured output.**

***Index Terms*— input saturation; sector-bounded nonlinearity; anti-windup compensation;**

I. Introduction

Saturation is probably the first class of non-linearity which any engineer who deals with the control of practical systems has to cope with. The components of physical processes that are most concerned by saturation phenomenon are indubitably the actuators. Since all actuators have their own physical limits (power, bandwidth,...), physical processes that are actuated can not be driven with any dynamic. But, often because of cost reasons, designers choose the components of a process in order to satisfy a nominal behavior a little bit far from the real use of the system, so that actuators are quickly faced with saturation. This happen more often than expected. Moreover, the first knowledges learned in the automatic control practitioners' community is often the linear control theory which assume that a controller can deliver a control signal of any magnitude. It is now well known that unconstrained linear plants controlled by an efficient linear controller when working in the linear operating range lead, in the better case, to poor performances when actuators are saturated, or it leads to instability in the worst case.

One of the most popular approaches allowing to deal with the saturation of the actuators is the one implementing anti-windup compensators. Roughly speaking, anti-windup compensator is a kind of controller of the pre-existing linear controller that is designed in order to stabilize the closed-loop system when it works in the saturated operating range, while ensuring some performance properties.

Among these sought properties, the so-called input/output-$\mathscr{L}_2$-gain performance index has received a great attention since a small couple of decades, especially since it has been mathematically rigourously formulated in [2]. There are LMI based methods translating quadratic stability as well as circle criterion or Popov criterion like in [3], [4], [5], [6], [7], [8], [9]. There are other approaches, such as those based on coprime factorization as in [10] or [11], that have been also proposed.

In this article, some previous results concerning the design of dynamic but plant-order anti-windup compensators are revisited. It concerns more precisely results of article [1] whose main idea is based on adding a new constraint to the derivative of the quadratic Lyapunov function in order to force the closed-loop dead-zone signal to be less than a given threshold. This constraint is translated mathematically as a narrowed version of the sector-bounded condition. The results proposed in this paper generalizes those in [1] to linear plants that may contain a direct feedthrough term relating the bounded control inputs to the measured outputs. It appears that the extension of results in [1] is not trivial, whereas several applications need results that explicitly take into direct feedthrough terms. For example, in active vibration control of thin mechanical structures piezo-actuated, the finite dimension linear models derived from a *Finite Element* analysis of the mechanical Partial Derivative Equations [12] always contain feedthrough terms in the finite dimension analysis model as well as in the synthesis one, in order to correct the static response and the anti-resonance frequencies of the inputs-outputs transfer functions [13]. A well known way to overcome the presence of direct feedthrough term is to filter the inputs (or outputs) with low-pass filters that are strictly proper and having a large bandwidth, decoupled with the plant's dynamic. This an obvious trick that has an appealing side, but the price to pay becomes non ridiculous when dealing with MIMO systems, leading to plant matrices of higher size and then complicating the numerical resolution of the problem.

The paper is organized as follows: in Section II are presented the notations used throughout this paper. They are voluntary taken similar to those in [1] to help the reader in the comparison. In Section III, the problem is addressed and in Section IV, the main results are exposed. A numerical illustration of these results on a practical application to active vibration control problem is presented shortly at the end. One can see [14] and [15] to have a more complete idea about this application.

II. Notations

$\mathbb{R}$ stands for the set of real numbers. Let k,l be some nonzero integers, $\mathbb{R}^k$ is the set of vectors of dimension k. $\mathbb{R}^{k\times l}$ is the vector space of rectangular matrices of dimension $k \times l$ with real coefficients. $\mathbb{1}_k$ is the identity matrix of dimension $k \times k$, 0 is the rectangular zero matrix of appropriate dimension. When needed for a better understanding, $\mathbf{0}_{k\times l}$

Work supported by grants from CNRS, DIGITEO and Région Ile-de-France. S. Tliba is with *Laboratoire des Signaux & Systèmes* UMR8506, Univ Paris-Sud, CNRS and SUPELEC, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France. sami.tliba@lss.supelec.fr

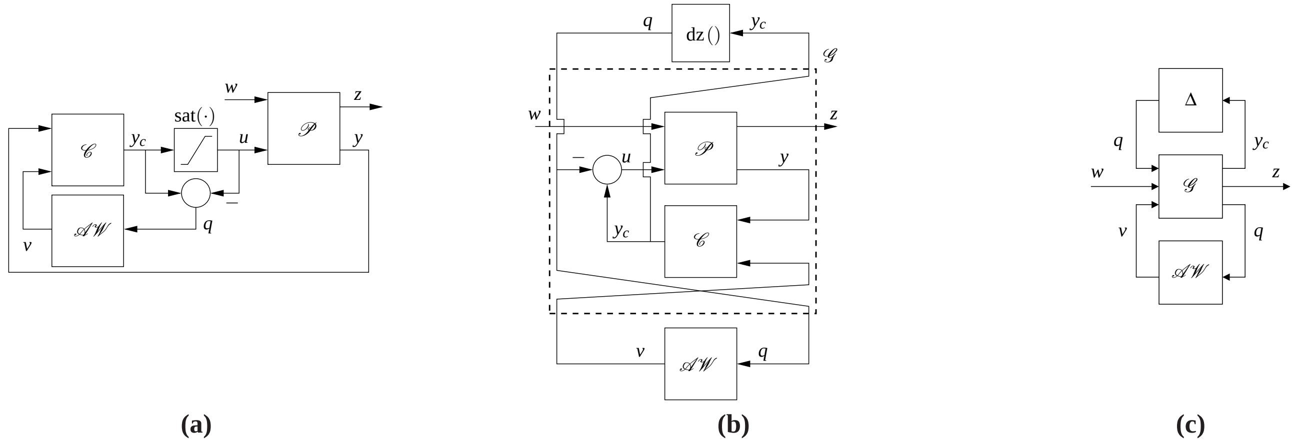

Fig. 1. (a): Non-linear closed loop interconnection with anti-windup compensation; (b): Non-linear closed loop interconnection recasted into a robust performance standard form; (c): Compacted robust performance standard form.

will denote the matrix of zeros with k rows and l columns. The set of real symmetric square $n\times n$ matrices is denoted $\mathbb{S}^{n\times n}$ and $\mathbb{S}_+^{n\times n}$ is those of positive definite matrices. Let $a,b\in\mathbb{R}$, sect$[a,b]$ denotes the conic sector defined by the set $\{(x,y)\in\mathbb{R}\times\mathbb{R}/(y-ax)(y-bx)\leq 0\}$ (see for example Fig. 2). The inverse of square matrix M is denoted M^{-1} and the Moore-Penrose pseudo-inverse is denoted $M^\dagger$. The space of square integrable functions is denoted $\mathscr{L}_2$ and the $\mathscr{L}_2$-norm of a signal $x\in\mathbb{R}^n$, of $\mathscr{L}_2$, is denoted $\|x\|_2 := \left(\int_0^\infty x^T(\tau)x(\tau)d\tau\right)^{1/2}$.

Throughout this paper, given a signal $u\in\mathbb{R}$, the usual saturation function will be considered, which is defined as

$$\mathrm{sat}(u) := \begin{cases} u, & |u|\leq u^{max}, \\ \mathrm{sign}(u)\,u^{max}, & |u|>u^{max}, \end{cases} \tag{1}$$

where u^{max} is the saturation threshold. The dead-zone function $\mathrm{dz}()$ is defined using the $\mathrm{sat}()$ function as $q=\mathrm{dz}(u) := u-\mathrm{sat}(u)$. Those notations are extended to vectorial signal $u\in\mathbb{R}^m$ by applying them at each component.

III. Problem definition

A. Closed-loop interconnection features

Given an LTI plant $\mathscr{P}$ described by

$$\mathscr{P}\begin{cases} \dot{x}_p = A_px_p + B_{p,w}w + B_{p,u}u \\ z = C_{p,z}x_p + D_{p,zw}w + D_{p,zu}u \\ y = C_{p,y}x_p + D_{p,yw}w + D_{p,yu}u \end{cases} \tag{2}$$

and its stabilizing controller $\mathscr{C}$ in state-space form, with appropriate matrices:

$$\mathscr{C}\begin{cases} \dot{x}_c = A_cx_c + B_cy + v_1 \\ y_c = C_cx_c + D_cy + v_2 \end{cases} \tag{3}$$

where $x_c\in\mathbb{R}^{n_c}$ is the controller state vector, $x_p\in\mathbb{R}^{n_p}$ is the plant state vector, $y\in\mathbb{R}^{n_y}$ is the measured output, $u\in\mathbb{R}^{n_u}$ is the control input of the plant and $y_c\in\mathbb{R}^{n_u}$ is the unconstrained linear controller's output. The controlled output is $z\in\mathbb{R}^{n_z}$ and the disturbance input is $w\in\mathbb{R}^{n_w}$. The input vector $v^T=\begin{bmatrix}v_1^T & v_2^T\end{bmatrix}$, $v\in\mathbb{R}^{n_v}$, $v_1\in\mathbb{R}^{n_c}$, $v_2\in\mathbb{R}^{n_u}$ and $n_v=n_u+n_c$, corresponds to additional inputs available that will be supplied by the sought external anti-windup compensator. These inputs are intended to modify the dynamic behavior of the controller $\mathscr{C}$ in (3) when working in the saturated operating range, in order to stabilize the input-saturated closed-loop and bring some performances during the saturation of the control input.

In this paper the only assumptions made concerning the plant are:

- (A1) the triple $(A_p,B_{p,u},C_{p,y})$ is stabilisable and detectable,
- (A2) the linear closed-loop interconnection of $\mathscr{P}$ and $\mathscr{C}$, *i.e.* when $u=y_c$ and $v=0$, is stable and well-posed.

Assumption (A1) is necessary and sufficient to allow for the plant stabilization by dynamic output feedback [16]. Assumption (A2) means that the linear controller has been successfully designed so that asymptotic stability is guaranteed. Of course, such controller should ensure some linear performance requirements. On the contrary of [1], nothing is required concerning the direct feedthrough term $D_{p,yu}$ as it will be shown, except that the closed loop is well posed. Moreover, the assumption referred as (A2) of [1], *i.e.* the one concerning full row rank of matrices $\begin{bmatrix}B_{p,u}^T & D_{p,zu}^T\end{bmatrix}$ and $\begin{bmatrix}C_{p,y} & D_{p,yw}\end{bmatrix}$, is useless here.

B. Anti-windup compensator design problem

Given an integer $n_{aw}\in\mathbb{N}$, we seek for a n_{aw}-order linear anti-windup *compensator* $\mathscr{AW}$ of input $q=\mathrm{dz}(y_c) := y_c-\mathrm{sat}(y_c)$, output $v^T=\begin{bmatrix}v_1^T & v_2^T\end{bmatrix}$ and of order n_{aw}, with dynamic:

$$\mathscr{AW}\begin{cases} \dot{x}_{aw} = A_{aw}x_{aw} + B_{aw}q \\ v = C_{aw}x_{aw} + D_{aw}q \end{cases} \tag{4}$$

where $x_{aw}\in\mathbb{R}^{n_{aw}}$. The anti-windup compensator is interconnected with the system following the structure depicted in Fig. 1 (a) and there equivalent form 1 (b) and 1 (c).

Let $\mathscr{G}$ be the closed-loop interconnection of $\mathscr{P}$ and $\mathscr{C}$ obtained by setting $u=y_c$ (see Fig. 1 (b)), where the state vector is denoted $x^T := \begin{bmatrix}x_p^T & x_c^T\end{bmatrix}$, $x\in\mathbb{R}^n$, $n:=n_p+n_c$.

Consider the closed-loop interconnection defined by the lower LFT[1] $\mathscr{T} := \mathscr{F}_l(\mathscr{G}, \mathscr{AW})$ between the linear closed-loop plant $\mathscr{G}$ and the anti-windup compensator, as depicted in Fig. 1 (b) and Fig. 1 (c), describing the closed-loop relation between w and z under the internal loop describing the sector-bounded uncertainty $q = \mathrm{dz}(y_c)$. This interconnection will be referred as the *anti-windup closed-loop system*. Given the corresponding closed-loop state vector $x_{cl}^T := \begin{bmatrix} x^T & x_{aw}^T \end{bmatrix}$, $x_{cl} \in \mathbb{R}^{n_{cl}}$ where $n_{cl} = n + n_{aw}$, its state-space model is:

$$\mathscr{T} \begin{cases} \dot{x}_{cl} = A_{cl}x_{cl} + B_{0,cl}q + B_{1,cl}d \\ y_c = C_{0,cl}x_{cl} + D_{00,cl}q + D_{01,cl}d \\ z = C_{1,cl}x_{cl} + D_{10,cl}q + D_{11,cl}d \end{cases} \tag{5}$$

$$q = \mathrm{dz}(y_c) \tag{6}$$

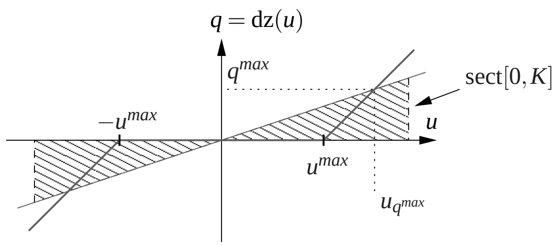

Fig. 2. Dead-zone nonlinearity and modified sector bounds

This paper addresses the problem of designing such anti-windup compensators that verify the following properties:

Property 1: Given a performance level $\gamma > 0$, a bound $q_i^{\max}$ on each dead-zone signal $q_i = \mathrm{dz}(y_{c_i})$ (see Fig. 2). Let $K = \mathrm{diag}\{k_1, k_2, \ldots, k_{n_u}\} > 0$ where $k_i = \frac{q_i^{max}}{q_i^{max} + u_i^{max}} \leq 1$. The anti-windup closed-loop system of Fig. 1 (a) ensure the following specification:

1) the anti-windup closed-loop system is well-posed,
2) there exists a quadratic Lyapunov function $V(x_{cl}) = x_{cl}^T P x_{cl}$ with $P \in \mathbb{S}_+^{n_{cl} \times n_{cl}}$ and a constant $\varepsilon > 0$ such that its derivative along the anti-windup closed-loop system's trajectories (5) satisfies:

$$\begin{cases} \dot{V}(x_{cl}) + \varepsilon x_{cl}^T x_{cl} + \dfrac{1}{\gamma} z^T z - \gamma w^T w \\ \qquad\qquad + 2q^T W (K y_c - q) < 0 \end{cases} \tag{7}$$

for any $W = \mathrm{diag}\{W_1, W_2, \ldots, W_{n_u}\} > 0$.

Remark 1: In the class of anti-windup compensators that satisfy item 1), according to $\mathscr{S}$-procedure results [18], the second item gives a sufficient condition for quadratic internal stability of the anti-windup closed-loop system, for finite $\mathscr{L}_2$-gain from w to z and for a bound q^{max} on dead-zone signals.

Remark 2: The last term of (7) comes from the modified sector-bounded condition with respect to the scalar product defined in Definition 1 of [7]. It is obvious that if $q^T W (K y_c - q) < 0 \ \forall t \geq 0$ holds for any diagonal matrix $W > 0$, then $(q, y_c) \in \mathrm{sect}[0, K]$ will be satisfied (*i.e.* $q^T (K y_c - q) < 0$ so that $q \leq q^{max}$) and then $(q, y_c) \in \mathrm{sect}[0, \mathbb{1}_{n_u}]$ will also be verified.

[1]LFT stands for Linear Fractional Transformation. See [17].

The following analysis result is due to [1]:

***Theorem** 1 (Robust performance analysis):* Given a performance level $\gamma > 0$ and the unconstrained closed-loop linear system $\mathscr{T}$ with initial condition $x_{cl}(0) = 0$ subjected to the sector-bounded uncertainty (6) , if there exist $P \in \mathbb{S}_+^{n_{cl} \times n_{cl}}$ and a diagonal matrix $W > 0$ such that

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & \left\{ \begin{array}{c} P B_{0,cl} \\ + C_{0,cl}^T K W \end{array} \right\} & P B_{1,cl} & C_{1,cl}^T \\ \left\{ \begin{array}{c} B_{0,cl}^T P \\ + W K C_{0,cl} \end{array} \right\} & \left\{ \begin{array}{c} W K D_{00,cl} \\ + D_{00,cl}^T K W \\ -2W \end{array} \right\} & W K D_{01,cl} & D_{10,cl}^T \\ B_{1,cl}^T P & D_{01,cl}^T K W & -\gamma \mathbb{1}_{n_w} & D_{11,cl}^T \\ C_{1,cl} & D_{10,cl} & D_{11,cl} & -\gamma \mathbb{1}_{n_z} \end{bmatrix} < 0, \tag{8}$$

then, the unconstrained closed-loop linear system $\mathscr{T}$ is robustly stable against the sector-bounded uncertainty $q = \mathrm{dz}(y_c)$, the dead-zone signal is bounded $q < q^{max}$ and the $\mathscr{L}_2$-gain condition $\|z\|_2 < \gamma \|w\|_2$ is satisfied.

C. Modified closed-loop features

In order to use results presented in [1] for a plant with a non zero direct feedthrough term $D_{p,yu}$, define the following output

$$\tilde{y} := y - D_{p,yu} u \tag{9}$$

$$= C_p x_p + D_{p,yw} w \tag{10}$$

leading to the strictly proper plant $\widetilde{\mathscr{P}}$ of output $\tilde{y}$, as assumed by the referred paper. The corresponding controller for an equivalent closed-loop is then

$$\widetilde{\mathscr{C}} \begin{cases} \dot{x}_c = \tilde{A}_c x_c + \tilde{B}_c \tilde{y} + \tilde{M}_c q + \tilde{v}_1 \\ y_c = \tilde{C}_c x_c + \tilde{D}_c \tilde{y} + \tilde{N}_c q + \tilde{v}_2 \end{cases} \tag{11}$$

where $\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{n_c} & B_c D_{p,yu} \Delta_c^{-1} \\ \mathbf{0}_{n_u \times n_c} & \Delta_c^{-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and

$$\Delta_c = \mathbb{1}_{n_u} - D_c D_{p,yu}. \tag{12}$$

Matrices of the equivalent controller $\widetilde{\mathscr{C}}$ to be connected to the strictly proper plant $\widetilde{P}$ are defined below:

$$\begin{array}{rcl} \tilde{A}_c & := & A_c + B_c D_{p,yu} \Delta_c^{-1} C_c \\ \tilde{B}_c & := & B_c \left(\mathbb{1}_{n_y} + D_{p,yu} \Delta_c^{-1} D_c \right) \\ \tilde{C}_c & := & \Delta_c^{-1} C_c \\ \tilde{D}_c & := & \Delta_c^{-1} D_c \\ \tilde{M}_c & := & -B_c D_{p,yu} \left(\mathbb{1}_{n_u} + \Delta_c^{-1} D_c D_{p,yu} \right) \\ \tilde{N}_c & := & -\Delta_c^{-1} D_c D_{p,yu} \end{array} \tag{13}$$

Compared to results presented in [1], new matrices $\tilde{M}_c$ and $\tilde{N}_c$ appeared and their presence can not be ignored in the main results of anti-windup synthesis. Indeed, these matrices are those making dead-zone signal q entering in the augmented controller state equations $\widetilde{\mathscr{C}}$ in (11).

Now let write the matrices of the closed-loop system $\mathscr{G}$, corresponding to the feedback of the controller $\mathscr{C}$ with the strictly proper plant, as depicted in Fig. 1 (b).

$$
\mathscr{G} \quad \begin{cases} \dot{x} = \tilde{A}x + \tilde{B}_0 q + \tilde{B}_1 d + \tilde{B}_2 v \\ y_c = \tilde{C}_0 x + \tilde{D}_{00} q + \tilde{D}_{01} d + \tilde{D}_{02} v \\ z = \tilde{C}_1 x + \tilde{D}_{10} q + \tilde{D}_{11} d + \tilde{D}_{12} v \\ q = \mathbb{1}_{n_u} q \end{cases} \tag{14}
$$

$$
q = \mathrm{dz}\left(y_c\right) \tag{15}
$$

where $x \in \mathbb{R}^n$, $n := n_p + n_c$ and the new closed-loop matrices are:

$$
\begin{aligned}
\tilde{A} &:= \begin{bmatrix} A_p + B_{p,u}\tilde{D}_c C_{p,y} & B_{p,u}\tilde{C}_c \\ \tilde{B}_c C_{p,y} & \tilde{A}_c \end{bmatrix}; \\
\tilde{B}_0 &:= \begin{bmatrix} B_{p,u}\left(\tilde{N}_c - \mathbb{1}_{n_u}\right) \\ \tilde{M}_c \end{bmatrix}; \; \tilde{C}_0 := \begin{bmatrix} \tilde{D}_c C_{p,y} & \tilde{C}_c \end{bmatrix}; \\
\tilde{B}_1 &:= \begin{bmatrix} B_{p,w} + B_{p,u}\tilde{D}_c D_{p,yw} \\ \tilde{B}_c D_{p,yw} \end{bmatrix}; \; \tilde{D}_{01} := \tilde{D}_c D_{p,yw}; \\
\tilde{B}_2 &:= \begin{bmatrix} \mathbf{0}_{n_p \times n_c} & B_{p,u}\Delta_c^{-1} \\ \mathbb{1}_{n_c} & B_c D_{p,yu}\Delta_c^{-1} \end{bmatrix}; \; \tilde{D}_{02} := \begin{bmatrix} \mathbf{0}_{n_u \times n_c} & \Delta_c^{-1} \end{bmatrix}; \\
\tilde{C}_1 &:= \begin{bmatrix} C_{p,z} + D_{p,zu}\tilde{D}_c C_{p,y} & D_{p,zu}\tilde{C}_c \end{bmatrix}; \\
\tilde{D}_{10} &:= D_{p,zu}\left(\tilde{N}_c - \mathbb{1}_{n_u}\right); \; \tilde{D}_{12} := \begin{bmatrix} \mathbf{0}_{n_e \times n_c} & D_{p,zu}\Delta_c^{-1} \end{bmatrix}; \\
\tilde{D}_{11} &:= D_{p,zw} + D_{p,zu}\tilde{D}_c D_{p,yw}; \; \tilde{D}_{00} := \tilde{N}_c;
\end{aligned} \tag{16}
$$

Following the procedure described both in [16] and [1], the anti-windup compensator parameters are gathered into the following matrix variable

$$
\Theta := \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} \in \mathbb{R}^{(n_p+n_c)\times(n_p+n_c)} \tag{17}
$$

so that the matrices $A_{cl}, B_{k,cl}, C_{j,cl}, D_{jk,cl}$ with $\{j,k\} \in \{0,1\}$ can be written linearly with respect to Θ as

$$
\begin{bmatrix} A_{cl} & B_{0,cl} & B_{1,cl} \\ C_{0,cl} & D_{00,cl} & D_{01,cl} \\ C_{1,cl} & D_{10,cl} & D_{11,cl} \end{bmatrix} = \begin{bmatrix} \mathscr{A} & \mathscr{B}_0 & \mathscr{B}_1 \\ \mathscr{C}_0 & \tilde{D}_{00} & \tilde{D}_{01} \\ \mathscr{C}_1 & \tilde{D}_{10} & \tilde{D}_{11} \end{bmatrix} + \begin{bmatrix} \mathscr{P}_1^T \\ \mathscr{P}_2^T \\ \mathscr{P}_3^T \end{bmatrix} \Theta \begin{bmatrix} \mathscr{Q}_1 & \mathscr{Q}_2 & \mathscr{Q}_3 \end{bmatrix} \tag{18}
$$

where

$$
\begin{bmatrix} \mathscr{A} & \mathscr{B}_0 & \mathscr{B}_1 \\ \mathscr{C}_0 & \tilde{D}_{00} & \tilde{D}_{01} \\ \mathscr{C}_1 & \tilde{D}_{10} & \tilde{D}_{11} \end{bmatrix} := \left[\begin{array}{cc|c|c} \tilde{A} & 0 & \tilde{B}_0 & \tilde{B}_1 \\ 0 & 0 & 0 & 0 \\ \hline \tilde{C}_0 & 0 & \tilde{D}_{00} & \tilde{D}_{01} \\ \hline \tilde{C}_1 & 0 & \tilde{D}_{10} & \tilde{D}_{11} \end{array}\right] \tag{19}
$$

$$
\begin{bmatrix} \mathscr{P}_1 & \mathscr{P}_2 & \mathscr{P}_3 \end{bmatrix} := \left[\begin{array}{cc|c|c} 0 & \mathbb{1}_{n_{aw}} & 0 & 0 \\ \tilde{B}_2^T & 0 & \tilde{D}_{02}^T & \tilde{D}_{12}^T \end{array}\right] \tag{20}
$$

$$
\begin{bmatrix} \mathscr{Q}_1 & \mathscr{Q}_2 & \mathscr{Q}_3 \end{bmatrix} := \left[\begin{array}{cc|c|c} \mathbb{1}_{n_{aw}} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{n_u} & 0 \end{array}\right] \tag{21}
$$

Remark 3: One can notice that equation (18) (and the related ones (19)-(20)-(21)) of the anti-windup closed-loop system is written with the new matrices in (16) and its expression does not differ at all to the corresponding one in [1]. Thus, one might be tempted to replace basically matrices of (16) into the feasibility and anti-windup construction results of [1]. But this amounts to underestimate the importance of the direct feedthrough term $D_{p,yu}$ in the matrices composing the real LMI conditions for non-strictly proper plants.

This remark is essential for the details of the proof of the next result about anti-windup feasibility.

IV. Main results

A. Anti-windup feasibility

Theorem* 2 *(Feasibility): Given a non-strictly proper LTI plant $\mathscr{P}$ and a stabilizing controller $\mathscr{C}$. Consider assumptions (A1)-(A2) and the equivalent strictly proper plant $\widetilde{\mathscr{P}}$ and its associated controller $\widetilde{\mathscr{C}}$. Given real scalars $0 < k_i \leq 1$, $i = 1, 2, \ldots, n_u$ and given a scalar $\gamma > 0$, a bound on the desired $\mathscr{L}_2$-norm of the closed-loop system from input(s) w to output(s) z (see Fig. 1 (c) for notations).

$$
\begin{bmatrix}
\left\{ \begin{array}{c} A_p R_{11} + R_{11} A_p^T \\ -B_{p,u}\left(\left(\tilde{N}_c - \mathbb{1}_{n_u}\right)V + V\left(\tilde{N}_c^T - \mathbb{1}_{n_u}\right)\right)B_{p,u}^T \end{array} \right\} & \left\{ \begin{array}{c} R_{11} C_{p,z}^T \\ +2B_{p,u}\left(\mathbb{1}_{n_u} - K^{-1}\right) V D_{p,zu}^T \end{array} \right\} & B_{p,w} \\
\left\{ \begin{array}{c} C_{p,z} R_{11} \\ +2D_{p,zu} V\left(\mathbb{1}_{n_u} - K^{-1}\right) B_{p,u}^T \end{array} \right\} & \left\{ \begin{array}{c} -\gamma \mathbb{1}_{n_z} \\ 2D_{p,zu}\left(\mathbb{1}_{n_u} - K^{-1}\right) V D_{p,zu}^T \end{array} \right\} & D_{p,zw} \\
B_{p,w}^T & D_{p,zw}^T & -\gamma \mathbb{1}_{n_w}
\end{bmatrix} < 0 \tag{22}
$$

$$
\begin{bmatrix} \tilde{A}^T S + S\tilde{A} & S\tilde{B}_1 & \tilde{C}_1^T \\ \tilde{B}_1^T S & -\gamma \mathbb{1}_{n_w} & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma \mathbb{1}_{n_z} \end{bmatrix} < 0 \tag{23}
$$

$$
\begin{bmatrix} R_{11} & \begin{bmatrix} \mathbb{1}_{n_p} & 0 \end{bmatrix} \\ \begin{bmatrix} \mathbb{1}_{n_p} \\ 0 \end{bmatrix} & S \end{bmatrix} \geq 0 \tag{24}
$$

If there exist positive definite matrices $R_{11} \in \mathbb{S}_+^{n_p \times n_p}$, $S \in \mathbb{S}_+^{n \times n}$, and a diagonal matrix $V = \mathrm{diag}\{v_1, v_2, \ldots, v_{n_u}\} > 0$ satisfying the convex inequalities (22)-(23)-(24), then there exist an anti-windup compensator $\mathscr{AW}$ of order $n_{aw} = n_p$ that robustly stabilizes the closed-loop system $\mathscr{G}$ with respect to the modified sector-bounded uncertainty $\mathrm{sect}[0, K]$.

Proof: The proof of this theorem follows closely the corresponding one in [1] except that the relations (18) to (21) are now considered in the development of the expression (8) of Theorem 1. These relations take explicitly into account the matrices of the modified controller $\widetilde{\mathscr{C}}$ in (11), associated with the strictly proper plant $\widetilde{\mathscr{P}}$, whose matrices are defined in (13). The details are then voluntary omitted. ■

Remark 4: In comparison to the results presented in [1] for the strictly proper case, (22)-(23) and the related inequalities in [1] present differences in almost all elements where matrices (in (16)) of the closed-loop system $\mathscr{T}$ in (5) appear. It is especially noticeable for the extra-terms introduced to deal with the modified sector-bounded conditions that are in the elements $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$. It emphasizes the importance to develop general conditions since those in [1] could not be simply extended to non-strictly proper plant. The converse is true since when assuming $D_{p,yu} = 0$, we recover results of [1].

Remark 5: Because of the narrow relation between the $\mathscr{L}_2$-gain and the $\mathscr{H}_\infty$ norm [18], it is then interesting to compare both approaches to emphasize the advantage of the present one. Indeed, one should note that γ is bounded by the $\mathscr{H}_\infty$ norm of the closed-loop system $\mathscr{F}_l\left(\mathscr{G}, \mathscr{AW}\right)$, *i.e.*

from input(s) w to output(s) z (see Fig. 1 (c) for notations). Generally, in the $\mathscr{H}_\infty$ control approach of the robust performance problem associated with the standard form Fig. 1 (b) and (c), when specifying $\gamma \leq 1$, all the requirements on the closed-loop will be fulfilled if the associated feasibility problem has a solution. These requirements are

- quadratic internal stability,
- closed-loop performance in the $\mathscr{H}_\infty$-norm sense expected in the channel $w \to z$,
- robustness of this closed-loop against the unstructured uncertainty Δ coming from the modified sector-bounded input nonlinearity.

In the case of "sub-optimal" $\mathscr{H}_\infty$-anti-windup-compensators, *i.e.* when $\gamma > 1$, one must pay attention about this situation which could happen frequently. This case means that one or more of the requirements are not fully met. This can be damageable for the anti-windup compensated closed-loop system if it concerns the requirement of robustness against the sector-bounded non-linearity. In that case, global as well as local stability could even not be ensured.

Remark 6: In practice, it may happen that a small γ does not lead to the existence of a solution, especially when the required performances are very important, most than those "allowed" in presence of the sector-bounded nonlinearity. So, under the same assumptions than in Theorem 2, one can seek for the "best" feasible γ, read the minimum one. Since the LMIs (22)-(23)-(24) are all convex with respect to the variable γ, the feasibility convex problem of Theorem 2 can easily be turned into the following optimization problem

$$\min_{R_{11},S,V} \gamma \quad \textit{subject to} \quad (22)\ (23)\ (24)$$

Remark 7: Condition (24) is devoted to the obtention of a plant-order anti-windup compensator. Indeed, this condition comes from the more general non-convex condition which states that a n_{aw}th-order anti-windup compensator has to verify instead (see [7] and more generally [16]):

$$\begin{bmatrix} R & \mathbb{1}_n \\ \mathbb{1}_n & S \end{bmatrix} \geq 0 \tag{25}$$

$$\operatorname{rank}\left(R - S^{-1}\right) \leq n_{aw} \tag{26}$$

The above non-convex conditions are satisfied for a full-rank anti-windup compensator, *i.e.* when $n_{aw} = n = n_p + n_c$. The only tractable reduced-order case are the plant-order case $n_{aw} = n_p$ (condition (24)) and the static case ($n_{aw} = 0$). For this last, by imposing that $R = S^{-1}$, the feasibility conditions are easily derived: condition (22) remain unchanged; condition (23) is slightly modified by scaling is on the left and on the right by the block-diagonal matrix $\operatorname{diag}\{R, \mathbb{1}_{n_w}, \mathbb{1}_{n_s}\}$; condition (24) is removed since it is obviously verified.

B. Anti-windup compensator construction

In this subsection, the same methodology as in [1] for the anti-windup compensator construction is used, *i.e.* when using explicit formulaes for the construction of $\mathscr{AW}$. This approach is based on results published in [19]. Once again, when considering a non-strictly-proper plant, formulas change and it is necessary to write them clearly in order to make them compatible with all cases. So, the following theorem is proposed to be used as a method for the anti-windup compensator's matrix construction:

***Theorem** 3 (Anti-windup compensator construction):* Given the solutions R_{11}, S, γ and V of the feasibility (or optimization) convex problem of Theorem 2. Let $W = V^{-1}K^{-1} = K^{-1}V^{-1}$, $H^T = \left[\mathbb{1}_{n_{aw}}\ 0_{n_{aw}\times(n-n_{aw})}\right]$ and consider the following decomposition $MN^T = \mathbb{1}_n - RS$ where $M, N \in \mathbb{R}^{n\times n_{aw}}$, $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22}\end{bmatrix}$ with $R_{12} = \left[\mathbb{1}_{n_p}\ 0\right] S^{-1} \begin{bmatrix} 0 \\ \mathbb{1}_{n_c}\end{bmatrix}$, $R_{22} = \begin{bmatrix} \mathbb{1}_{n_c} \\ 0 \end{bmatrix} S^{-1} \left[0\ \mathbb{1}_{n_c}\right] = R_{22}^T$. Then, an n_{aw}th-order anti-windup compensator, $n_{aw} \geq n_p$, can be obtained by using the following method:

(i) Compute a feasible $\hat{D}_{aw} \in \mathbb{R}^{n_v\times n_c}$ such that

$$\underbrace{\begin{bmatrix} \left\{\begin{matrix} WK\left(\tilde{D}_{00} + \tilde{D}_{02}\hat{D}_{aw}\right) \\ -2W \\ +\left(\tilde{D}_{00}^T + \hat{D}_{aw}^T\tilde{D}_{02}^T\right)KW \end{matrix}\right\} & WK\tilde{D}_{01} & \left\{\begin{matrix} \tilde{D}_{10}^T \\ +\hat{D}_{aw}^T\tilde{D}_{12}^T \end{matrix}\right\} \\ \tilde{D}_{01}^T KW & -\gamma\mathbb{1}_{n_w} & \tilde{D}_{11}^T \\ \left\{\begin{matrix} \tilde{D}_{10} \\ +\tilde{D}_{12}\hat{D}_{aw} \end{matrix}\right\} & \tilde{D}_{11} & -\gamma\mathbb{1}_{n_z} \end{bmatrix}}_{=-\Pi} < 0 \tag{27}$$

(ii) Compute the least-square solutions of the following equations for $\hat{B}_{aw} \in \mathbb{R}^{n\times n_u}$, $\hat{C}_{aw} \in \mathbb{R}^{n_v\times n_{aw}}$

$$\left[\begin{array}{c|ccc} 0 & \mathbb{1}_{n_u} & 0 & 0 \\ \hline \mathbb{1}_{n_u} & & & \\ 0 & & -\Pi & \\ 0 & & & \end{array}\right] \left[\begin{array}{c} \hat{B}_{aw} \\ \hline \\ ? \\ \\ \end{array}\right] = \left[\begin{array}{c} 0_{n_u\times n} \\ \hline \tilde{B}_0^T S + WK\tilde{C}_0 \\ \tilde{B}_1^T S \\ \tilde{C}_1 S \end{array}\right] \tag{28}$$

$$\left[\begin{array}{c|ccc} 0 & \tilde{D}_{02}^T KW & 0 & \tilde{D}_{12}^T \\ \hline WK\tilde{D}_{02} & & & \\ 0 & & -\Pi & \\ \tilde{D}_{12} & & & \end{array}\right] \left[\begin{array}{c} \hat{C}_{aw} \\ \hline \\ ? \\ \\ \end{array}\right] = \left[\begin{array}{c} \tilde{B}_2^T H \\ \hline \left\{\begin{matrix} WK\tilde{C}_0 RH + \\ \hat{D}_{aw}^T\tilde{B}_2^T H + \tilde{B}_0^T H \end{matrix}\right\} \\ \tilde{B}_1^T H \\ \tilde{C}_1 RH \end{array}\right] \tag{29}$$

and the matrix $\hat{A}_{aw} \in \mathbb{R}^{n\times n_{aw}}$ as

$$\hat{A}_{aw} = -\tilde{A}^T H - X(\hat{B}_{aw})\Pi^{-1}Y(\hat{C}_{aw}, \hat{D}_{aw}) \tag{30}$$

where

$$X(\hat{B}_{aw}) := \left[S\tilde{B}_0 + \hat{B}_{aw} + \tilde{C}_0^T KW\ \ S\tilde{B}_1\ \ \tilde{C}_1^T\right] \tag{31}$$

$$Y(\hat{C}_{aw}, \hat{D}_{aw}) := \begin{bmatrix} \left\{\begin{matrix} (\tilde{B}_0^T + \hat{D}_{aw}^T\tilde{B}_2^T)H \\ +WK\tilde{C}_0 RH + WK\tilde{D}_{02}\hat{C}_{aw} \end{matrix}\right\} \\ \tilde{B}_1^T H \\ \tilde{C}_1 RH + \tilde{D}_{12}\hat{C}_{aw} \end{bmatrix} \tag{32}$$

(iii) Compute Θ, the variable containing the original matrices of the anti-windup compensator in (17) by the algebraic relation:

$$\begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} = \begin{bmatrix} N & S\tilde{B}_2 \\ 0_{n_v \times n_{aw}} & \mathbb{1}_{n_v} \end{bmatrix}^{\dagger} \left(\begin{bmatrix} \hat{A}_{aw} & \hat{B}_{aw} \\ \hat{C}_{aw} & \hat{D}_{aw} \end{bmatrix} - \begin{bmatrix} S\tilde{A}RH & 0_{n \times n_u} \\ 0_{n_v \times n_{aw}} & 0_{n_v \times n_u} \end{bmatrix} \right) \begin{bmatrix} M^T H & 0_{n_{aw} \times n_u} \\ 0_{n_u \times n_{aw}} & \mathbb{1}_{n_u} \end{bmatrix}^{\dagger} \quad (33)$$

Remark 8: Given a solution $\hat{D}_{aw}$ of (27), an alternative for the calculation of $\hat{B}_{aw}$ (respectively of $\hat{C}_{aw}$) would be to solve LMI (34) (respectively LMI (35)). Indeed, it can be shown, following results in [19], that solutions of (28) and (29) are those leading to the most uniformly negative definite solution of the below LMIs

$$\tilde{A}^T S + S\tilde{A} + X(\hat{B}_{aw})\Pi^{-1} X(\hat{B}_{aw})^T < 0 \quad (34)$$

$$H^T \left(\tilde{A}R + R\tilde{A}^T\right) H + H^T \tilde{B}_2 \hat{C}_{aw} + \hat{C}_{aw}^T \tilde{B}_2^T H + Y(\hat{C}_{aw}, \hat{D}_{aw})^T \Pi^{-1} Y(\hat{C}_{aw}, \hat{D}_{aw}) < 0, \quad (35)$$

that can be solved as they are to obtain a solution for $\hat{B}_{aw}$ and $\hat{C}_{aw}$.

V. Simulation Example

Consider the problem of active vibration control of a flexible beam equipped piezoelectric sensor and actuator that was introduced in [14] or in [15]. The reduced-order model is of order 6 and is not strictly proper. A linear 6th-order $\mathscr{H}_\infty$ controller with pole-placement constraint has been designed to meet all requirements of vibrations' attenuation and robustness against unmodelled dynamic. The other simulation parameters are the same than those in [14] and [15]. In order to compare results in [1] with those presented in this paper, we consider the synthesis of a plant-order dynamic anti-windup compensator for each case, *i.e.* the case of strictly proper plant by using results of [1] and the case of non-strictly proper plant by using results of this paper. For the non-strictly proper case, the augmented plant is of order 7 whereas for the strictly proper case, it is of order 8 because the input is filtered by a first-order low-pass filter to eliminate the direct feedthrough term. The frequency cut-off is set to $10^5 Hz$ in order to be completely decoupled with the plant dynamics. For both approaches, we set a bound q^{max} on the dead-zone signal to $6200V$ corresponding to $k = 0.992$.

Using the same optimization parameters for both cases, for the strictly proper case, the anti-windup compensator is of order 8 and gives a $\mathscr{L}_2$-gain $\gamma_{sp} = 8.55$, whereas for the non-strictly proper case, the anti-windup compensator is of order 7 and gives a $\mathscr{L}_2$-gain $\gamma_{nsp} = 12.56$. The proposed approach seems to lost a little bit of performance in $\mathscr{L}_2$-gain sense while winning on the anti-windup compensator's complexity. Both approaches satisfy the condition of boundedness on the dead-zone signal. To illustrate that, Fig. 3 propose a non-linear time simulation in closed-loop for the case of unconstrained control, then for the case of saturating control with and without anti-windup compensator.

VI. Conclusion

In this paper, some LMI conditions has been proposed to address the design problem of anti-windup compensators for exponentially unstable and not strictly proper plants with bounded inputs, that achieve quadratic stability, $\mathscr{L}_2$-gain performance and an upper bound on the dead-zone signal. Results proposed in this paper generalizes those in [1] to whatever linear plant, strictly proper or not. A quite conclusive simulation on a practical application has been proposed in order to compare both of these results.

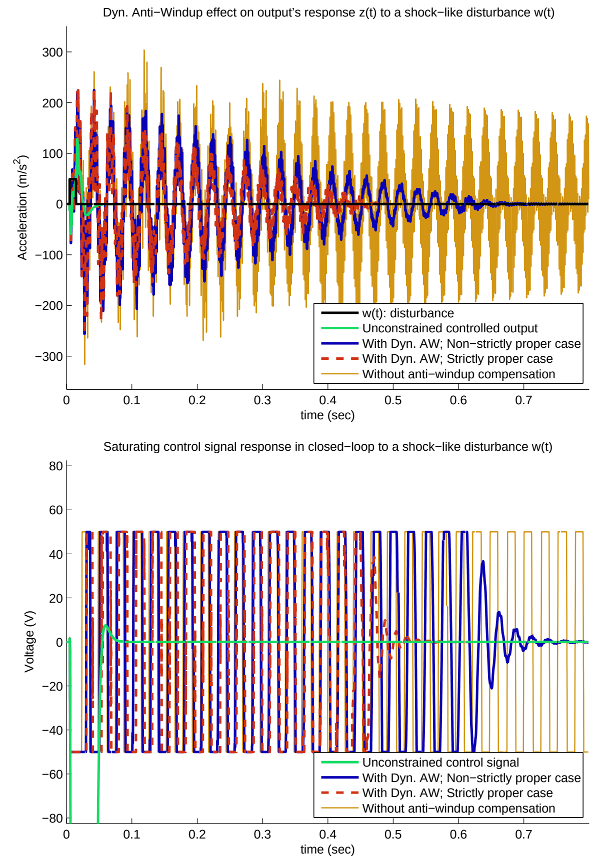

Fig. 3. Comparison of the anti-windup closed-loop behavior responses

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