Reliable Network Coding

Lana Iwaza, Marco Di Renzo, and Michel Kieffer*

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Network coding [2, 42], and more specifically random linear network coding [39, 13, 26, 28], is a powerful tool for delivering information across a network. Random coding techniques may be implemented in a distributed way within network elements, independently of the structure of the network. In [28], it has been shown that the max-flow capacity of the network can be reached with probability exponentionally approaching one with the size of the Galois field in which the random coding operations are performed. This work has led to a number of practical schemes such as COPE, ANC, MIXIT, and MORE, etc. [35, 34, 33].

Nevertheless, network coding is very sensitive to transmission errors, packet losses, and corrupted packets which are intentionally injected by malicious nodes. Recombinations carried out by each node lead to a progressive contamination of the set of clean packets by the erroneous ones, which makes the decoding impossible at the receiver side. On the other hand, even in the absence of errors, losses of packets lead to an insufficient number of packets at the receiver side, making the use of the already received packets impossible.

Error correcting network coding techniques aim at protecting packets from transmission errors, form erreneous packets, and/or losses. Error correcting network coding techniques introduce a certain level of *redundancy* and are similar in principle to classical error correcting codes. We can distinguish between two families of codes. The codes introduced in [8, 62] both focus on network coding and the introduction of redundancy. These codes require an *a priori* knowledge of the architecture of the network and the way in which network coding is carried out, see Section 1 for further details. These results are extended to the framework of random network coding in [28, 4], see Section 2. The techniques introduced in [31, 38, 51, 1] exploit the fact that, in the absence of errors, random network coding preserves the space vector spanned by the transmitted packets. The proposed robust network codes have properties that are relatively independent from the way the network coding is carried out, see Section 3.

*Lana Iwaza, Marco Di Renzo, and Michel Kieffer work at Laboratoire des Signaux et Systèmes - CNRS - SUPELEC - Univ Paris-Sud, 3 rue Joliot-Curie, F-91192 Gif-sur-Yvette cedex `iwaza@lss.supelec.fr, marco.direnzo@lss.supelec.fr, kieffer@lss.supelec.fr`. Michel Kieffer is on leave at LTCI - CNRS - Télécom ParisTech, 46 rue Barrault, F-75013 Paris

Joint decoding techniques exploit the existing redundancy in the communication networks [17]. In the case of *joint channel-network decoding* [25, 53], temporal or spatial diversity or the presence of channel codes [23, 37] are used to combat the noise introduced by the communication channels, in particular wireless channels, see Section 4. *Joint source-network decoding* allows the recovery of all or part of the initial packets, even in the presence of an insufficient number of received packets, by exploiting the correlation between transmitted data packets, see Section 5. Therefore, these techniques provide a certain robustness against packet loss.

1 Coherent network error correction codes

The notations and the content of this section are largely inspired by [63, 58, 62]. For this type of network error correcting codes, the topology of the network as well as the considered network code are assumed to be known by each destination node [59, 9].

A communication network is described by a directed acyclical graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. A link $e = (i, j) \in \mathcal{E}$ represents a channel linking the nodes $i \in \mathcal{V}$ and $j \in \mathcal{V}$. The set of links emerging from a node $i \in \mathcal{V}$ is written as $O(i)$, the set of paths converging at i is written as $I(i)$. A multicast network is a triple $(\mathcal{G}, s, \mathcal{T})$, where $\mathcal{G}$ is a network, $s \in \mathcal{V}$ is the source and $\mathcal{T}$, the set of destination nodes. We assume that $I(s) = \emptyset$, $O(t) = \emptyset$ for every $t \in T$. Let $n_s = |O(s)|$. Subsequently, $\mathbb{F}$ represents the Galois field with q elements.

The source node s encodes the message to transmit as a row vector $\mathbf{x} = [x_1, \dots, x_{n_x}] \in \mathbb{F}^{n_x}$ called a *codeword*. The set of codewords is written as $\mathcal{C}$. Each component of $\mathbf{x}$ is therefore sent on one of the links of $O(s)$. An error vector $\mathbf{z} \in \mathbb{F}^{|\mathcal{E}|}$ allows us to describe the errors introduced by the links in the network. If we denote $\bar{f}_e$ and f_e as the input and output of the link e and if an error z_e is introduced on the link $e \in \mathcal{E}$, then $f_e = \bar{f}_e + z_e$. For every subset of links $\rho \in \mathcal{E}$, we introduce the two vectors $\mathbf{f}_\rho = [f_e, e \in \rho]$ and $\bar{\mathbf{f}}_\rho = \left[\bar{f}_e, e \in \rho\right]$. A code for the network $\mathcal{G}$ is therefore defined by a set of codewords $\mathcal{C} \subset \mathbb{F}^{n_s}$ and a family of local coding functions $\left\{\bar{\beta}_e, e \in \mathcal{E} \backslash O(s)\right\}$, with $\bar{\beta}_e : \mathbb{F}^{|I(\text{source}(e))|} \rightarrow \mathbb{F}$ such that

$$\bar{f}_e = \bar{\beta}_e \left(F_{I(\text{source}(e))}\right) \tag{1}$$

and where $\text{source}(e)$ indicates the node from which e emerges. Assume that the destination node t receives the vector $\mathbf{u}_t = (u_e,\ e \in I(t))$. An iterative application of (2) allows to express $\mathbf{u}_t$ as a function of $\mathbf{x}$ and of the error vector $\mathbf{z}$

$$\mathbf{u}_t = F_{st}(\mathbf{x}, \mathbf{z}). \tag{2}$$

where $F_{st}(\mathbf{x}, \mathbf{z})$ represents the set of network coding operations taking place between the source s and destination t. In the case of coherent network codes, the structure of $F_{st}(\mathbf{x}, \mathbf{z})$ is assumed to be known at the decoder and is used to perform the estimation of $\mathbf{x}$ from $\mathbf{u}_t$. In order to characterize the error correction capacity of a network code, it is necessary to introduce the notion of distance between codewords [62]. For that purpose, consider the set of vectors that can be received by the node t when the source transmits a codeword $\mathbf{x}$ and the network introduces an error vector $\mathbf{z}$ with a Hamming weight $w_H(\mathbf{z})$ less than c

$$\Phi_t(\mathbf{x}, c) = \{F(\mathbf{x}, \mathbf{z}) \text{ st } w_H(\mathbf{z}) \leqslant c\}. \tag{3}$$

It is possible to deduce from $\Phi_t(\mathbf{x}, c)$ a pseudo-distance between two codewords $\mathbf{x}$ and $\mathbf{y}$ emitted by the source

$$D_t(\mathbf{x}, \mathbf{y}) = \min\{c_1 + c_2 \text{ st } |c_1 - c_2| \leqslant 1 \text{ and } \Phi_t(\mathbf{x}, c_1) \cap \Phi(\mathbf{x}, c_2) \neq \emptyset\} \tag{4}$$

and a minimal distance for the network code at the node t

$$d_{\min,t} = \min\{D_t(\mathbf{x}, \mathbf{y}), \mathbf{x} \neq \mathbf{y}\}. \tag{5}$$

The decoder seeking the minimum weight error vector (maximum likelihood decoder if all code words have same probability) can therefore be constructed in the following way. First, we search for the set $\mathcal{P}$ of pairs $(\mathbf{x}, \mathbf{z})$ satisfying (2). In the sub-set $\mathcal{P}_w \subset \mathcal{P}$ of the pairs $(\mathbf{x}, \mathbf{z})$ whose Hamming weight $\mathbf{z}$ is minimal, if all the pairs have same $\mathbf{x}$, then the error is said to be *correctable* and $\mathbf{x}$ is the estimation of the transmitted message. If this is not the case, the error is not *correctable*. It has been shown in [57] that the correction capacity of a network code (with a decoder that searches for the minimum weight error vector) is $\lfloor (d_{\min} - 1)/2 \rfloor$, where $\lfloor \cdot \rfloor$ indicates the rounding towards $-\infty$. In the case of linear network codes, the functions $\bar{\beta}_e$ are linear and for every $e \in \mathcal{E} \backslash O(s)$, we have

$$\bar{f}_e = \sum_{e' \in \mathcal{E}} \beta_{e',e} F_{e'} \tag{6}$$

where $\beta_{e',e}$ is the local coding coefficient of the node e' towards the node e. Using (6), [39] has shown that (2) can be written as

$$\mathbf{u}_t = \mathbf{x}\mathbf{F}_{s,t} + \mathbf{z}\mathbf{F}_t, \tag{7}$$

where $\mathbf{F}_{s,t}$ and $\mathbf{F}_t$ can be deduced from (2) and are perfectly known. In the case of linear network codes, (4) becomes

$$D_t(\mathbf{x}, \mathbf{y}) = \min\{c \text{ st } (\mathbf{x} - \mathbf{y}) \mathbf{F}_{s,t} \in \Phi_t(c)\}, \tag{8}$$

with

$$\Phi_t(c) = \left\{\mathbf{z}\mathbf{F}_t, \, \mathbf{z} \in \mathbb{F}^{|\mathcal{E}|}, \; w_H(\mathbf{z}) \leqslant c\right\}, \tag{9}$$

the set of messages received when the zero code word is sent. The main bounds in terms of error correction codes have been extended to network codes in [59, 9, 58] such as the Hamming, the Singleton, and the Gilbert-Varhsamov bound, as well as in [7] for the Plotkin and Elias bounds. For the Hamming and Singleton bounds

$$d_{\min} = \min_{t \in \mathcal{T}} d_{\min,t} \tag{10}$$

and

$$n = \min_{t \in \mathcal{T}} \text{maxflow}\,(s,t)\,. \tag{11}$$

In the case of a network code for which $\text{rank}\,(\mathbf{F}_{s,t}) = r_t$ and $d_{\min,t} > 0$, the Hamming bound may be written as

$$|\mathcal{C}| \leqslant \min_{t \in \mathcal{T}} \frac{q^{r_t}}{\sum_{i=0}^{\tau_t} \begin{pmatrix} r_t \\ i \end{pmatrix} (q-1)^i}, \tag{12}$$

with $\tau_t = \lfloor (d_{\min,t} - 1)/2 \rfloor$. The Singleton bound becomes

$$|\mathcal{C}| \leqslant q^{r_t - d_{\min,t} + 1} \tag{13}$$

for every node t, see [58]. The Singleton bound (13) allows to extend the notion of MDS (for *Maximum Distance Separable*) codes to network codes [60]. A network code where the Singleton bound is reached is said to be MDS. It is optimal in the sense that it exploits all the redundancy in the network error correcting code. A code construction method enabling the Singleton bound (13) to be reached has been proposed in [58]. The technique consists of first constructing the local coding coefficients which ensure that the rank of matrices $\mathbf{F}_{s,t}$ is always sufficient. This can be done using the Jaggi-Sanders algorithm [32]. The codewords are then generated so that there is sufficient distance between them regardless of which destination node t is considered. The associated decoding algorithms are presented in [59, 9]. These techniques will be described in further details in the rest of this chapter. See also [47, 62] as well as [4] for further details on this type of codes.

2 Codes for non-coherent networks, random codes

The random network codes proposed in [26, 28] can be seen as a practical solution to network coding which can easily adapt to variations in the network topology since they are decentralized. In the case of random coding, the matrices $\mathbf{F}_{s,t}$ and $\mathbf{F}_t$ introduced in (7) are random. While it is possible to deduce $\mathbf{F}_{s,t}$ from the received packet headers (assuming that they have not been corrupted), $\mathbf{F}_t$, on the other hand, cannot be easily deduced. In the absence of transmission errors, the probability that a destination node $t \in \mathcal{T}$ is not capable of decoding the message received can be expressed as a function of the rank of $\mathbf{F}_{s,t}$

$$P_e^{(t)} = \Pr\left(\text{rank}(\mathbf{F}_{st}) < n_x\right). \tag{14}$$

The probability that at least one of the destination nodes is incapable of decoding the received message is deduced from (14)

$$P_e = \Pr\left(\exists t \in \mathcal{T} \text{ st } \text{rank}(F_{st}) < n_x\right), \tag{15}$$

see [28]. If c_t denotes the min-cut capacity between s and t, then $\delta_t = c_t - n_x$ corresponds to the redundancy at t. The probability of errors at the receiver t is therefore bounded as follows

$$P_e^{(t)} \leqslant 1 - \sum_{i=n_x}^{n_x+\delta_t} \binom{n_x+\delta_t}{i} \left(1-p-\frac{1-p}{q}\right)^{Li} \left(1-\left(1-p-\frac{1-p}{q}\right)^{L}\right)^{n_x+\delta_t-i} \tag{16}$$

where L indicates the length of the longest path between s and t and p is the link erasure probability. When the links are perfectly reliable ($p=0$), (16) becomes

$$P_e^{(t)} \leqslant 1 - \sum_{i=0}^{\delta_t} \binom{C_t}{i} \left(1-\frac{1}{q}\right)^{L(C_t-i)} \left(1-\left(1-\frac{1}{q}\right)^{L}\right)^{i}. \tag{17}$$

In the presence of errors, the results of [62] briefly presented in Section 1 can be extended. However, for a given code, the minimum distance $d_{\min,t}$ introduced in (5) becomes a random variable $D_{\min,t}$. Once the code $\mathcal{C}$ is fixed, the distance $d_{\min,t}$ will dependent on the (random) elements of $\mathbf{F}_{st}$. A partial characterization of $D_{\min,t}$ has been proposed in [4]

$$\Pr\left(D_{\min,t} < \delta_t + 1 - d\right) \leqslant \frac{\binom{|\mathcal{E}|}{\delta_t - d}\binom{d+|\mathcal{J}|+1}{|\mathcal{J}|}}{(q-1)^{d+1}}, \tag{18}$$

where $\mathcal{J} \subset \mathcal{E}$ is the set of internal nodes in the network. This result allows to deduce the probability of existence of an MDS code according to the size q of the Galois field in which the coding operations take place, see [4] for further details.

3 Codes for non-coherent networks, subspace codes

The network coding error correcting techniques proposed in [38, 51] are very different from the ones previously introduced. A non-coherent network model is considered, where neither the coder nor the decoder need to know the topology of the network nor the way in which combinations of packets are carried out. This work is motivated by the fact that, in the absence of errors, network coding preserves the vector space spanned by the transmitted packets. The coding operation is carried out via the transmission of a vector space inside a set of possible vector spaces (which represents the set of codewords). A destination node must identify the vector sub-space belonging to the code found to be the closest (in a sense to be defined) to the vector space spanned by the received packets. The received vector space can be different from the one that has been transmitted, depending on the packet losses, transmission errors or erroneous packets deliberately injected by malicious nodes.

3.1 Principle of subspace codes

In this approach, the transmission of information from the source s to a destination node t is conveyed by the injection into the network of a vector sub-space $V \subset \mathbb{F}^n$ and by the reception of a sub-space $U \subset \mathbb{F}^n$. Let $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_{n_s}\}$, with $\mathbf{x}_i \in \mathbb{F}^n$, be the set of vectors (data packets) injected by the source s and forming a base of V. In the absence of errors, $t \in T$ receives a set of packets $\mathbf{u} = \{\mathbf{u}_1, \ldots, \mathbf{u}_{n_t}\}$ formed by linear combinations of $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n_s}\}$, such that $\mathbf{u}_j = \sum_{i=1}^{n_s} h_{ji}\mathbf{x}_i$, where the h_{ji} are random coefficients of $\mathbb{F}$. The effect of potential transmission errors is modeled by the introduction of *packets* of errors $\mathbf{z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_{n_z}\}$ throughout the network. Since these packets can be injected into any link or node in the network, at receiver side, one gets

$$\mathbf{u}_j = \sum_{i=1}^{n_s} h_{ji}\mathbf{x}_i + \sum_{k=1}^{n_z} g_{jk}\mathbf{z}_k, \tag{19}$$

where the $g_{jk} \in \mathbb{F}$ are again random. In matrix form, one obtains

$$\mathbf{u} = H\mathbf{x} + G\mathbf{z}. \tag{20}$$

The model (19) is close to (7), but in (7), symbols belonging to $\mathbb{F}$ are transmitted while in (19) packets are sent through the network. In (7), $\mathbf{F}_{s,t}$ and $\mathbf{F}_t$ are perfectly known when the network structure and the network coding operations are known, which is not the case with the coefficients h_{ji} and g_{jk} (this is why we consider here non-coherent network codes). With this type of model, the aim of the receiver cannot be to precisely identify $\mathbf{x}$, but rather to identify the vector sub-space V spanned by the vectors of $\mathbf{x}$, based on the knowledge of the vector sub-space U created by the elements of $\mathbf{u}$. To introduce the notion of subspace codes, we consider a vector space W of dimension n on $\mathbb{F}$, for example$\mathbb{F}^n$. $\mathcal{P}(W)$ is the set of all the vector sub-spaces of W. The dimension of a sub-space $V \in \mathcal{P}(W)$ is written as $\dim(V)$. We can show that [38] for every $A \in \mathcal{P}(W)$ and $B \in \mathcal{P}(W)$,

$$d(A, B) = \dim(A + B) - \dim(A \cap B) \tag{21}$$

is a distance between vector sub-spaces. A subspace code is therefore a subset of $\mathcal{C} \subset \mathcal{P}(W)$. A *codeword* of $\mathcal{C}$ is a vector sub-space of $\mathcal{C}$. The minimum distance of $\mathcal{C}$ is the minimum distance between two distinct codewords while using the distance (21)

$$d_{\min}(\mathcal{C}) = \min_{X,Y \in \mathcal{C},\ X \neq Y} d(X, Y). \tag{22}$$

The maximum dimension of the code words of $\mathcal{C}$ is $\ell(\mathcal{C}) = \max_{X \in \mathcal{C}} \dim(X)$. When the dimension of all the codewords of $\mathcal{C}$ is the same, then the code is of constant dimension. Assume that a codeword $V \in \mathcal{C}$ is sent by the source, that U is received by a destination $t \in \mathcal{T}$, it is possible to describe the behavior of the network as

$$U = H_k(V) \oplus Z \tag{23}$$

with $k = \dim(U \cap V)$, $H_k(V)$ is a sub-space of V with dimension k such that $H_k(V) \cap V = 0$. This type of model illustrates the impact of network coding and the introduction of errors in terms of operations on vector sub-spaces. With this model, the network introduces $\rho = \dim(V) - k$ cancellations and $n_z = \dim(Z)$ errors. In this case, [38] shows that if $2(n_z + \rho) < d_{\min}(\mathcal{C})$, then a decoder with a minimum distance allows getting V from U. A generalization of the Singleton bound is proposed for these codes [38]. A construction of codes on sub-spaces similar to Reed-Solomon codes allowing the Singleton bound to be reached as well as a decoding algorithm with minimum distance for this family of codes is detailed in [38], emphasizing constant dimension codes.

3.2 Recent developments

This research has lead to a number of recent developments. Constant dimension codes are studied in [55] and applied to network coding by demonstrating that Steiner structures are optimal constant dimension codes. Johnson-type bounds are also calculated. In [20], several new codes and bounds exploiting the distance between sub-spaces (21) are explored. In [51], a wide class of constant dimension codes is studied, a new distance considering the rank metrics is introduced. Codes associated with this metric are introduced and an effective decoding algorithm for this family is proposed. Several constant dimension codes are introduced in [40] with a larger number of codewords than in the case of the previously examined codes. Performance bounds as well as construction methods for the code family introduced in [38] are proposed in [1]. An analysis of the geometric properties of the codes using rank metric is carried out in [21]. The lower and upper bounds of the cardinality of codes of given rank are evaluated which enables an analysis of the performance of these codes. In [19], a new multi-level approach examining the construction of subspace codes is presented. The authors show that the codes proposed in [38] represent a specific case of the proposed family of codes. A Gilbert-Varshamov bound relative to the codes constructed in [52] is introduced in [36], exploiting the injection metric. Finally, [11] studies the practical implantation of the codes introduced in [38]. The construction of these codes for small Galois fields and limited error correction capacity is feasible, and improves the network performance in terms of throughput.

4 Joint network-channel coding/ decoding

This section aims to show how, in a wireless context, the redundancy existing at the network level can help to improve channel decoding performance by performing joint network-channel decoding. This joint approach allows to reduce the number of packets lost due to transmission errors on wireless networks. This is achieved by using, on one hand, the network spatial diversity and on the other hand, the redundancy introduced by channel codes on the low layers of communication protocols. This research is motivated by [18], who highlights the limits of the coding approaches in which the network and the channel or the source and the network are separated. Studies carried out on canonical networks demonstrate that source-channel separation remains valid for some networks although this is not the case for network-channel separation. In [41], it is also shown that despite the fact that separation remains valid in some cases, a separate processing, for example source-network, results in higher costs, for example in terms of bandwidth or energy, than a joint treatment.

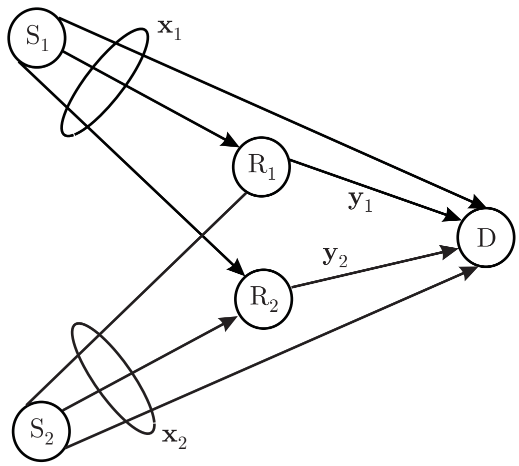

Figure 1: Wireless networks with two sources, two relays and one destination

Wireless networks are a privileged area of application for joint network-channel decoding techniques. In contrast to wired networks where lower layers of the protocol stack are supposed to provide error-free links, wireless networks provide packets which may be erroneous. Joint network-channel decoding techniques exploit the redundancy introduced by the network coding operations in order to improve the capacity of the channel code to correct transmission errors. Instead of focusing on guaranteeing an error-free transmission on each link, we are more interested in guaranteeing error-free decoding at the destination nodes. The latter use the data received from incoming links for decoding. In the presence of links providing a certain level of redundancy, error-free decodingis possible even if decoding at the level of each individual link is not possible. Joint network-channel decoding is therefore only useful when network coding introduces redundancy. The first practical application of this concept to networks with relays has been proposed in [25]. Iterative network-channel decoding methods for relay networks as well as for multiple access relay channels have been proposed in[24] and [25].

4.1 Principle

Consider a wireless network topology consisting of two sources S_1 and S_2, two intermediate relay nodes R_1 and R_2, and a destination node D, see Figure 1.

The sources generate two information messages $\mathbf{x}_1$ and $\mathbf{x}_2$ of k symbols each, and protect them using channel codes in order to obtain two independent packets with n symbols each, $\mathbf{p}_1$ and $\mathbf{p}_2$, which are then transmitted towards D. The relays receive the two packets, process them and retransmit them to D. To simplify, the links are assumed to be without errors and the communications are carried out on two orthogonal channels where mutual interference is negligible. As a result, D receives four packets from which it attempts to recover the information messages $\mathbf{x}_1$ and$\mathbf{x}_2$ sent by the sources. Assume that the two packets $\mathbf{p}_1$ and $\mathbf{p}_2$ can be expressed as a function of $\mathbf{x}_1$ and $\mathbf{x}_2$ as follows

$$\mathbf{p}_1 = \mathbf{x}_1 G_1 \text{ et } \mathbf{p}_2 = \mathbf{x}_2 G_2, \tag{24}$$

where G_1 and G_2 are two channel coding matrices with of dimension $k \times n$ with elements belonging to $\mathbb{F}$, a Galois field with q elements. The relay nodes R_1 and R_2 directly receive $\mathbf{p}_1$ and $\mathbf{p}_2$ and are therefore capable of decoding them to obtain $\mathbf{x}_1$ and $\mathbf{x}_2$, which are then re-encoded using a channel code and a network code to obtain

$$\mathbf{y}_1 = a_{11}\mathbf{x}_1 G_{11} + a_{12}\mathbf{x}_2 G_{12} \tag{25}$$

and

$$\mathbf{y}_2 = a_{21}\mathbf{x}_1 G_{21} + a_{22}\mathbf{x}_2 G_{22} \tag{26}$$

where the a_{ij} are network coding coefficients and where the matrices G_{ij} are channel code generator matrices used at the relay. As a result, D receives four packets $\mathbf{p}_1, \mathbf{p}_2, \mathbf{y}_1$ and $\mathbf{y}_2$ from which it has to estimate $\mathbf{x}_1$ and $\mathbf{x}_2$ transmitted by the source. By adopting a matrix notation, one obtains the following equations

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{y}_1 \\ \mathbf{y}_1 \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \\ a_{11}G_{11} & a_{12}G_{12} \\ a_{21}G_{21} & a_{22}G_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = G_{\text{joint}} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{27}$$

where G_{joint} represents the generator matrix for the joint network-channel code. From (27), one sees that channel and network codes can be considered as a unique code from the point of view of the network extremities and that the latter can be represented by a unique generator matrix G_{joint}. As a result, in the presence of transmission errors, the messages $\mathbf{x}_1$ and $\mathbf{x}_2$ can be decoded at the destination by directly exploiting G_{joint} or by the use of an iterative decoding method, see for example [23].

4.2 Recent developments

Several studies linked to joint network-channel decoding have been proposed in [24] and [25]. These studies have focused on relaxing some of the hypotheses introduced in [25], like assuming that the error correction is perfect between the source and the relays, see for example [5], [56], or [3]. Other results on joint network-channel decoding and more specifically code optimization can be found in [61].

5 Joint source-network coding/ decoding

Joint source-network coding and decoding enable all or some of the packets transmitted by the sources to be recovered in the presence of an insufficient number of received network-coded packets, by exploiting the existing or artificially introduced correlation between the transmitted data packets. These techniques also enable the distributed compression of correlated messages generated by geographically distributed sources.

Regarding the robustness against losses, or capacity variations on some of the links of the network, an alternative solution to the network coding techniques presented in Sections 1 to 3 consists of combining multiple description coding techniques [**?**], [22] and network coding. The aim is to exploit the redundancy introduced by these coding techniques in order to allow a progressive improvement of the quality of data reconstructed with the number of packets received at the receiver nodes [30]. An already existing correlation between data generated by the sources can also be exploited in order to obtain a scheme more robust to packet losses.

Regarding distributed compression, distributed source coding [**?**, **?**, 15] can perform separate compression of correlated sources and may be as effective (when there are no losses) as joint compression. This technique is interesting in the case of sensor networks where it is possible to perform effecient compression even in the absence of coordination between sensors [14, **?**]. This solution does not, however, allow to completely exploit the capacity of the network and assumes that the sensors have a precise estimate of the level of correlation between the data they produce. In this context, network coding is a natural solution for correlated data transmission on a network with diversity. The application of network coding for the compression of correlated sources has been proposed in [6, 27, 54] in the case of lossless coding. The proposed techniques provide effecient distributed algorithms which are capable of exploiting diversity whether at the source or the channel level. In the case of coding with losses, *compressed sensing* [16, 10] allows an approximate reconstruction of the source by exploiting its properties of compressibility using random combinations of its samples. Network coding techniques inspired by compressed sensing have been proposed in [50] using network codes on the real fields. However, the data taken from wireless sensor networks are, in general, quantized and network coding in the case of real fields is therefore questionable.

5.1 Exploiting redundancy to combat loss

In order to combat packet losses in the network, it is possible to exploit the redundancy existing in the data generated by the source(s). This redundancy may be introduced artificially, as in [30] or present naturally, as in [29].

5.1.1 Artificially introduced correlation

Two techniques for introducing redundancy are examined in [30]. The first involves a transformation matrix whose coefficients belong to $\mathbb{F}$ after quantization, see Figure 2.

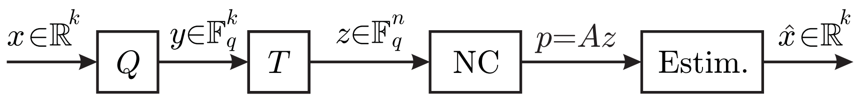

Figure 2: Joint source-network coding with redundancy introduced by transformation

The samples $\mathbf{x} \in \mathbb{R}^k$ generated by the source are quantized on q levels and are then transformed using a redundant transformation $T \in \mathbb{F}^{n \times k}$ with full rank k to obtain $\mathbf{z} = T\mathbf{y}$. As a result, there exists a matrix$D \in \mathbb{F}^{(n-k) \times n}$ with full rank $n - k$ such that

$$D\mathbf{z} = 0.$$

The elements of $\mathbf{z}$ are then transmitted in the network where the network coding operations, represented by the matrix A, are performed at the intermediate nodes. At the decoder side, the matrix

$$B = \begin{pmatrix} A \\ D \end{pmatrix}$$

is constructed from the received packets. If there exists a sub-matrix B' of B such that B' is full rank k, then the elements of $\mathbf{z}$ can be reconstructed by a simple Gaussian elimination. This approach provides a good robustness against packet losses with a decoding complexity similar to that of classical network coding.

The second technique introduces redundancy via a frame expansion [22] of data generated by the source, see Figure 3.

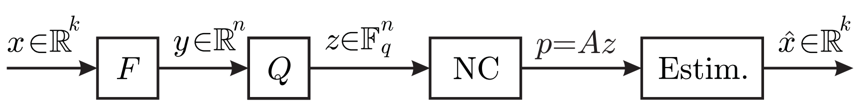

Figure 3: Joint source-network coding with redundancy introduced via a frame expansion

The samples $\mathbf{x} \in \mathbb{R}^k$ generated by the source are transformed using a frame expansion $F \in \mathbb{R}^{n \times k}$ in order to obtain $\mathbf{y} = F\mathbf{x} \in \mathbb{R}^n$. A frame on $\mathbb{R}^k$ is a set of $n > k$ vectors $\{\varphi_i\}_{i=1 \dots n}$ such that there is $B > 0$ and $C < \infty$ satisfying for every $\mathbf{x} \in \mathbb{R}^k$,

$$B \|x\|^2 \leq \sum_{i=1}^{n} \langle x, \varphi_i \rangle^2 \leq C \|x\|^2 , \tag{28}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product of $\mathbb{R}^k$. The correlated samples $\mathbf{y}$ are then quantized using a uniform quantizer with step size Δ with q levels to obtain a vector $\mathbf{z} \in \mathbb{F}^n$. The samples of $\mathbf{z}$ are placed in independent packets and transmitted in the network where they undergo network coding operations represented by a matrix A. When A has full rank n, an estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$ can be obtained from the received packets $\mathbf{p}$ by inverting the network coding matrix A, which provides an estimate

$$\widehat{\mathbf{z}} = Q^{-1}\left(A^{-1}\mathbf{p}\right), \tag{29}$$

of $\mathbf{z}$, with Q^{-1} being the quantizer reconstruction function. This estimate is then used to obtain an estimate of $\mathbf{x}$

$$\widehat{\mathbf{x}} = \left(F^T F\right)^{-1} F^T \widehat{\mathbf{z}}. \tag{30}$$

When not enough packets are received, the coding matrix A cannot be inverted. Since no unique estimate of $\mathbf{x}$ can be inferred from received packets $\mathbf{p} \in \mathbb{F}^m$, one selects the one with the minimal norm

$$\widehat{\mathbf{x}} = \arg\min \mathbf{x}^T \mathbf{x} \tag{31}$$

under the constraints

$$\begin{cases} \mathbf{p} = A\mathbf{z}, \\ \mathbf{y} = F\mathbf{x}, \\ \mathbf{z} \in \mathbb{F}^n \\ \mathbf{x} - \alpha\mathbf{z} - \beta \leqslant \Delta.\mathbf{1}, \\ -\mathbf{x} + \alpha\mathbf{z} + \beta \leqslant \Delta.\mathbf{1}. \end{cases} \tag{32}$$

In (32), the first constraint allows us to take into account the received packets, the second, the fact that the vector to be estimated have been transformed using a frame. The third expresses the fact that each component of $\mathbf{z}$ belongs to $\{0, \ldots, q-1\}$. The two latter constraints allow to take into account the bounded character of the quantization noise. This constrained optimization is difficult because it combines the real variables $\mathbf{x}$ and $\mathbf{z}$ with the components of $\mathbf{z}$ belonging to a Galois field. When the size of the Galois field q is prime, the network coding operations can be expressed in the ring of integers $\mathbb{Z}$ by introducing an additional vector $\mathbf{s} \in \mathbb{Z}^m$ in order to express the first constraint as

$$\mathbf{p} = A\mathbf{z} + q\mathbf{s}. \tag{33}$$

The solution for (31) under constraints (32) where the first constraint is replaced by (33) requires the resolution of a mixed integer quadratic optimization problem. This type of problem can be modeled with AMPL and solved with CPLEX. Estimation complexity here is much higher than that of a redundancy introduction technique using a transformation in a Galois field. However, a part of the quantization noise can be suppressed when a high number of packets is received.

5.1.2 Existing correlation

In this case, we assume that the source(s) generate correlated data $\mathbf{x} \in \mathbb{R}^k$, assumed, for example, to be the realization of a Gaussian vector $\mathbf{X}$ of mean zero and a non-diagonal covariance matrix Σ. A typical scenario corresponds to several sensors dispersed geographically and taking correlated measures x_i, $i = 1, \ldots, k$. These measures are quantized in order to obtain the samples $z_i \in \mathbb{F}^k$ transmitted on the network where they are coded. The effect of network coding is represented by a coding matrix A and a set of packets $\mathbf{p} = A\mathbf{z}$ is obtained at the collection point. A maximum *a posteriori* estimator of $\mathbf{z}$ from $\mathbf{p}$ is proposed in [29]

$$\widehat{\mathbf{z}} = \arg\max_{\mathbf{z}} P(\mathbf{z}|\mathbf{p}). \tag{34}$$

The Gaussian probability distribution and the correlation between samples of$\mathbf{x}$ are exploited to obtain a new mixed quadratic optimization problem modeled with AMPL and solved with CPLEX. Robustness against losses increase with the correlation between the components of $\mathbf{x}$.

5.2 Joint source-network coding

The aim of joint source-network coding in a network of sensors is to simultaneously collect and compress data in the network. This section examines the problem of joint source-network coding in presence of losses. We will consider a network of sensors described by a directed acyclical graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. Among the nodes of this graph, there are source nodes, $s \in \mathcal{S} \subset \mathcal{V}$, collector nodes $t \in \mathcal{T} \subset \mathcal{V}$ and intermediate nodes belonging to $\mathcal{E} \setminus (\mathcal{S} \cup \mathcal{T})$. Assume that there is a single collector node $|\mathcal{T}| = 1$, and that for every $s \in \mathcal{S}$, there is a path from s to t. The aim is to estimate in t the data$x_1, \ldots, x_n$ with $x_i \in \mathbb{R}$ and $n = |\mathcal{S}|$, exploiting the correlation between x_is, in order to minimize exchanges on the network. This problem is linked to robust network coding by the fact that the proposed collection scheme must eventually be robust to losses introduced by certain edges of the network.

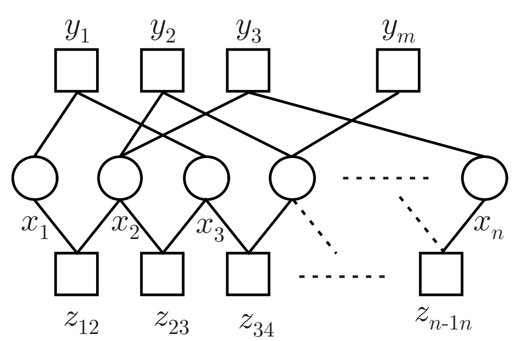

Figure 4: Bipartite graph associated with the joint source-network coding problem

The lossy coding method proposed in [43] assumes that the correlation between the x_i is known at t. The method consists of quantizing x_i at each source using a uniform quantizer on q levels. The quantized data are then sent on the network where they are network coded. The collector node t receives $m \leqslant n$ coded packets $y_1, \ldots, y_m$ from which it has to estimate $x_1, \ldots, x_n$. For this, it exploits the existing correlation between the x_is. It is possible to formally write a maximum *a posteriori* estimator of $\mathbf{x} = (x_1, \ldots, x_n)$ from $\mathbf{y} = (y_1, \ldots, y_m)$ as follows

$$\widehat{\mathbf{x}}_{\mathrm{MAP}} = \arg\max_{\mathbf{x}} p\left(\mathbf{x}|\mathbf{y}\right), \tag{35}$$

which allows us to take into account all the information that t has. However, the evaluation complexity of $\widehat{\mathbf{x}}_{\mathrm{MAP}}$ is exponential in the number of sensors and, even for small networks, an exact implantation of (35) is not feasible. It is however possible to represent the relations between components of $\mathbf{x}$ and $\mathbf{y}$ using a bipartite graph [49], where the variable nodes are x_i and the *check* nodes are y_j, corresponding to the received packets and the nodes z_k allow to account for the correlation between the components of $\mathbf{x}$, see Figure 4. Belief propagation algorithms [44, 49] can then be employed to obtain an estimate of the *a posteriori* marginals $p\left(x_i|\mathbf{y}\right)$ from which an *a posteriori* component by component approximation can be obtained. This estimation provides an approximate solution of (35). This technique is effecient when the data generated by sensors are highly correlated. Moreover, the matrix A representing the network coding operations (and which allows to deduce the links between the x_i and y_j in the bipartite graph) should be sufficiently sparse to allow convergence of the belief propagation algorithm. The way in which network coding must be carried out to ensure that A has the correct properties remains an unresolved problem to our knowledge.

6 Conclusion

This chapter has introduced several robust network coding techniques which aim is to cope with losses and errors introduced by links or nodes of the network. Coding techniques for coherent networks require the knowledge of the network structure and a centralized optimization of the way in which network coding is carried out. These techniques are well adapted to situations where the network structure is static. The main point of interest is that it is not necessary to introduce network coding coefficients into the packet headers passing through the network. Therefore, performance for this type of code is deterministic.

The previous techniques can be extended to a random network coding framework. This means that network coding is performed locally, in a distributed way, providing good adaptivity to variations in network topology. This type of code is therefore well adapted to mobile wireless networks. The disadvantage is that the performance of robust network code for coherent networks in this type of situation are described by random variables. It is not possible to guarantee a minimum distance for a robust network code. This requires transmitting the network coding coefficients in the packet headers which leads to an increase in the amount of data on the links.

The subspace codes represent an interesting alternative to the previously mentioned techniques, more specifically in a non-coherent network. These techniques are particularly effective against deletions or against erroneous packets injected deliberately by some of the nodes of the network, see [48]. These tools can be associated with joint network-channel decoding techniques. The joint exploitation of the redundancy of the channel code and of the network code can significantly reduce the number of packets that would be considered as erroneous if only one separate process would have been carried out. The cost, however, is an increase in the decoding complexity at the receiver.

Source-network coding techniques have a particular significance for improving network robustness to loss. These techniques also allow to have a more progressive improvement in the quality of decoded messages when the number of packets received by a destination increases. This can be interesting for transmitting multimedia contents, for example, in pair-to-pair networks. See [12, 46, 45] for further details. Finally, network coding can be seen as a highly significant tool for joint source-network coding since it allows an effective collection of data across sensor networks.

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