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***Abstract*— We consider continuous-time linear systems with additive disturbances and discrete-time measurements. First, we construct an observer, which converges to the state trajectory of the linear system when the maximum time interval between two consecutive measurements is sufficiently small and there are no disturbances. Second, we construct interval observers allowing to determine, for any solution, a set that is guaranteed to contain the actual state of the system when bounded disturbances are present.**

I. Introduction

Traditional state estimators, such as the Luenberger observer [21] or the Kalman filter [9], compute point estimates of the state from input-output data, possibly supplemented by an estimate of the dispersion of the possible values of the state around this point estimate. By contrast, *guaranteed state estimators* [5], [6], [18], also known as *set-membership estimators* [4], [26], compute sets guaranteed to contain the actual value of the state if some hypotheses on the state perturbation and measurement noise are satisfied.

Guaranteed state estimation can be traced back to the seminal work of F.C. Schweppe [32], [33]. His idea was recursively to compute ellipsoids *guaranteed* to contain the actual state. Of course, other types of containers than ellipsoids could and have been used, such as boxes [25], parallelotopes [7], zonotopes [1] or other limited-complexity polytopes. Matlab toolboxes implementing ellipsoidal or polytopic calculus are readily available [19], [36].

We consider in this paper a specific type of guaranteed state estimators for continuous-time linear models known as *interval observers*. Interval observers were introduced in [12] and extended and applied in many studies, see, for instance, [3], [22]–[24], [27], [29]–[31]. Typically, they bound the actual state between the solutions of two *deterministic* and possibly coupled dynamical systems, which form a *framer*. It is also required that the upper and lower bounds asymptotically converge to one another in the absence of state perturbation. The constructions of interval observers rely more or less directly on the notion of cooperative system [34], but they are not limited to this type of system, as explained in [22] and [23].

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Until recently, interval observers were designed for systems without output or with continuously measured outputs. This was a severe limitation as most, if not all, measurements are collected at discrete instants of time. In the pioneering contributions [10], [11], interval observers for nonlinear continuous-time systems using *discrete-time measurements* were introduced. The ideas of these contributions are: (*i*) to construct classical framers for the system under study, (*ii*) to reinitialize the framer at each measurement time, taking into account the current estimate and measured outputs.

The aim of the present work is to propose a new approach for building interval observers for continuous-time linear systems with discrete-time outputs. Our result differs significantly from those presented in [10], [11] because the interval observers of [10], [11] have discontinuous solutions whereas those proposed here have continuous solutions.

The key ideas of our approach can be decomposed into three steps. First, under a classical detectability condition, we construct an observer that would be exponentially stable if the outputs were available at all instants of time and no disturbances were acting. However, with discrete-time measurements and additive state disturbances, instability may occur (even if the disturbances are bounded, as we shall assume) in the sense that some trajectories may go to infinity when the time intervals between two measurements are larger than some threshold. We give conditions ensuring that this phenomenon does not occur. Second, we determine the error equation and transform the time-invariant part of it into a cooperative system through the possibly time-varying change of coordinates introduced in [23]. Third, using the observer and modified error equation, we construct an interval observer for the original system (with additive disturbances), which is discontinuous with respect to time. It admits continuous solutions and produces upper and lower bounds for the solutions which, in the absence of disturbances, converge exponentially to one another. It is worth mentioning that the observer on which we base our interval observer for systems with discrete-time measurements differs from those presented in [2], [8] and [13], which rely on an impulsive correction of the estimated solution that is carried out when a new measurement becomes available and thereby yield discontinuous solutions. Finally, we wish to point out that, in general, the classical interval observers that are valid for systems with continuous-time outputs do not even frame the solutions when the outputs are only available at discrete time instants. We will show that through a counter-example inspired from that in [24].

The paper is organized as follows. Section II is devoted to definitions, notation and a motivating counter-example. The system under study is introduced in Section III, where observer for it is constructed. A family of interval observers is proposed and studied in Section IV. Concluding remarks are drawn in Section V.

II. Notation, definitions and motivating counter-example

A. Basic notation and definitions

The Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimensions are denoted $||\cdot||$. For any integer k, the identity matrix of any dimension k is denoted by I and any $k\times n$ matrix, whose entries are all 0 is denoted by 0. Inequalities must be understood *componentwise* (partial order of $\mathbb{R}^r$), so for instance $W_a = (w_{a1},...,w_{ar})^\top \in \mathbb{R}^r$ and $W_b = (w_{b1},...,w_{br})^\top \in \mathbb{R}^r$ are such that $W_a \leq W_b$ if and only if, for all $i\in\{1,\ldots,r\}$, $w_{ai}\leq w_{bi}$. $\max\{A,B\}$ for two matrices $A=(a_{ij})$ and $B=(b_{ij})$ of the same dimensions is the matrix M such that each entry is $m_{ij} = \max\{a_{ij},b_{ij}\}$. A matrix $A=(a_{ij})\in\mathbb{R}^{n\times p}$ is nonnegative (resp. positive) if $a_{ij}\geq 0$, $i=1,...,n$, $j=1,\ldots,p$, (resp. $a_{ij}>0$, $i=1,...,n$, $j=1,...,p$). In this case, we will write $A\geq 0$ (resp. $A>0$). $A\succcurlyeq 0$ is used to express the fact that $A\in\mathbb{R}^{n\times n}$ is a symmetric matrix such that, for all vector $V\in\mathbb{R}^n$, $V^\top AV\geq 0$. A real square matrix is *cooperative* (or Metzler) if all its off-diagonal entries are nonnegative.

For a continuous function $\varphi:[-\tau,+\infty)\to\mathbb{R}^k$, for all $t\geq 0$, the function φ_t is defined by $\varphi_t(\theta)=\varphi(t+\theta)$ for all $\theta\in[-\tau,0]$.

The argument of the functions will be omitted or simplified when the context is such that no confusion can arise.

B. Interval observer: definition

For the sake of generality, we introduce a definition of interval observers for perturbed time-varying nonlinear systems with outputs containing noisy measurements of the state taken at discrete time instants. It differs from the one used in [23].

Definition 1: Consider a continuous-time dynamical system

$$\left\{\begin{array}{rcl} \dot{x}(t) &=& \mathscr{F}(t,x(t),u(t),w(t)), \\ y(t) &=& \mathscr{H}(x(t_i),w(t)), \text{ when } t\in[t_i,t_{i+1}), \end{array}\right. \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $w\in\mathbb{R}^\ell$ is the state perturbation, $y\in\mathbb{R}^p$ is the output, and $u\in\mathbb{R}^q$ is the input. The measurement times t_i form an increasing sequence with $t_0=0$ and are such that there are two constants $\varepsilon>0$, $\tau\geq\varepsilon$ such that $t_{i+1}-t_i\in[\varepsilon,\tau]$ for all integers $i\in\mathbb{N}$. $\mathscr{F}$ and $\mathscr{H}$ are functions of class C^1 with respect to x, u, w and $\mathscr{F}$ is piecewise-continuous with respect to t. The state perturbation w is piecewise-continuous and such that there exist known bounds $\overline{w}(t)=(w^-(t),w^+(t))\in\mathbb{R}^{2\ell}$, continuous and such that, for all $t\geq 0$,

$$w^-(t)\leq w(t)\leq w^+(t). \tag{2}$$

Then, the continuous-time dynamical system

$$\dot{z}(t)=\mathscr{A}(t,z(t),y(t),u(t),\overline{w}_t), \tag{3}$$

where $z\in\mathbb{R}^r$, where the function $\mathscr{A}$ is locally Lipschitz with respect to z, y, u and $\overline{w}_t$[1] on any bounded set of $\mathbb{R}\times\mathbb{R}^r\times\mathbb{R}^p\times\mathbb{R}^q\times C^0([-\tau,0])$ and piecewise-continuous with respect to t, associated with the initial condition $z_0=\mathscr{B}(s_0,x_0^+,x_0^-)\in\mathbb{R}^r$ and bounds for the solution x: $x^+=\mathscr{C}^+(t,z)$, $x^-=\mathscr{C}^-(t,z)$, with $\mathscr{C}^+$, $\mathscr{C}^-$, $\mathscr{B}$ Lipschitz continuous of appropriate dimension, is called an *interval observer* for (1) if

(i) for any vectors x_0, x_0^- and x_0^+ in $\mathbb{R}^n$ satisfying $x_0^-\leq x_0\leq x_0^+$ and any $u(\cdot)$, $w(\cdot)$, $\overline{w}(\cdot)$ bounded on any interval $[0,t)$, $t\geq 0$ such that (2) is satisfied, the solutions of (1), (3) with x_0, $z_0=\mathscr{B}(s_0,{x_0}^+,{x_0}^-)$ as initial conditions at $t=s_0$, denoted respectively $x(t)$ and $z(t)$, are defined for all $t\geq s_0$ and satisfy, for all $t\geq s_0$, the inequalities $x^-(t)\leq x(t)\leq x^+(t)$.

(ii) for any vectors x_0^- and x_0^+ in $\mathbb{R}^n$ satisfying $x_0^-\leq x_0^+$, the solution $z(t)$ of the system (3), with $\overline{w}(\cdot)$ identically equal to zero and with $z_0=\mathscr{B}(s_0,{x_0}^+,{x_0}^-)$ as initial condition at $t=s_0$, is such that $\lim\limits_{t\to+\infty}||x^+(t)-x^-(t)||=0$.

C. Counter example

In this section, we show, through a simple example, that classical continuous-time interval observers are not robust relative to sampling of their outputs, no matter how small the largest sampling interval is. This motivates the construction of interval observers for systems with sampled outputs to be presented in the subsequent sections.

Observe first that, for the one-dimensional system

$$\dot{x}(t)=x(t) \tag{4}$$

with the output $y(t)=x(t)$, the system

$$\left\{\begin{array}{rcl} \dot{z}^+(t) &=& -z^+(t)+2y(t), \\ \dot{z}^-(t) &=& -z^-(t)+2y(t), \end{array}\right. \tag{5}$$

associated with the bounds $x^+=z^+$, $x^-=z^-$ and the initial conditions $z_0^+=x_0^+$, $z_0^-=x_0^-$, is an interval observer [12]. Now, if, for all $t\in[t_i,t_{i+1})$, we replace $y(t)$ in (5) by $y(t_i)$, where $t_i=i\tau$ for all $i\in\mathbb{N}$ with τ any positive real number, we obtain, for all $t\in[t_i,t_{i+1})$,

$$\left\{\begin{array}{rcl} \dot{z}^+(t) &=& -z^+(t)+2y(t_i), \\ \dot{z}^-(t) &=& -z^-(t)+2y(t_i). \end{array}\right. \tag{6}$$

This system with the bounds $x^+=z^+$, $x^-=z^-$ and the initial conditions $z_0^+=x_0^+$, $z_0^-=x_0^-$ is *not* an interval observer for (4). Let us prove this. Let $\bar{z}^+=z^+-x$. We have, for all $t\in[t_i,t_{i+1})$,

$$\dot{\bar{z}}^+(t) \quad=\quad -\bar{z}^+(t)+2x(t_i)\left(1-e^{t-t_i}\right). \tag{7}$$

By considering the initial condition $x(s_0)=1$, $z^+(s_0)=1$, $s_0=0$ and integrating (7) over the interval $[t_0,t_1)=[0,\tau)$, we obtain that $\bar{z}^+(t)=2-e^{-t}-e^t$, for all $t\in[0,\tau)$. It follows that $\bar{z}^+(t)<0$ for all $t\in(0,\tau)$. This allows us to conclude.

[1] Note that $\overline{w}_t$ in (3) should not be confused with $\overline{w}(t)$ (see Section II-A for the meaning of the notation $\overline{w}_t$).

III. Observers for systems with discrete-time measurements

We now focus on the system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t)+\delta_1(t), \\ y(t) & = & Cx(t_i)+\delta_2(t), \text{ when } t\in[t_i,t_{i+1}), \end{array} \right. \quad (8)$$

where the matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times q}$, and $C\in\mathbb{R}^{p\times n}$ are constant, δ_1 and δ_2 are piecewise-continuous functions (which typically represent state perturbations and measurement noise) and t_i is an increasing sequence with $t_0=0$ and such that there exist two constants $\tau\geq\varepsilon>0$ for which

$$0<\varepsilon\leq t_{i+1}-t_i\leq\tau, \text{ for all } i\in\mathbb{N}. \quad (9)$$

In this section, we present an observer for continuous-time linear systems described by (8). This result will later be used to construct interval observers for the system (8). However, it is of interest for its own sake. To the best of our knowledge, it is new, in spite of its simplicity. Note that typically the disturbance $\delta_2(t)$ is constant over each interval $[t_i,t_{i+1})$ since $y(t)$ mostly represents discrete-time measurements. This fundamental case is covered by Theorems 1-2 below since δ_2 is assumed piecewise-continuous.

Two assumptions are needed.

Assumption 1: There exists a constant matrix $K\in\mathbb{R}^{n\times p}$ such that the matrix

$$H=A+KC \quad (10)$$

is Hurwitz. Moreover, $L=KC\neq 0$.
Assumption 1 ensures that there is a symmetric and positive definite matrix $S\in\mathbb{R}^{n\times n}$ such that the matrix inequality

$$H^\top S+SH\preccurlyeq -I \quad (11)$$

is satisfied.

Assumption 2: There exists a real number $a_*\in[||A||,+\infty)$, $a_*>0$ such that the constant τ introduced in (9) satisfies

$$\tau\in\left(0,\frac{1}{a_*}\ln\left(1+\frac{a_*}{2||L||}\right)\right], \quad (12)$$

and

$$\tau\in\left(0,\frac{1}{a_*}\ln\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{a_*}{2||SL||(2||L||+a_*)}}\right)\right], \quad (13)$$

where L is the matrix in Assumption 1 and S is a matrix satisfying (11).

We are ready to prove the following result:

Theorem 1: Assume that the system (8) satisfies Assumptions 1 and 2. Then the system defined by

$$\dot{\hat{x}}(t) \;=\; A\hat{x}(t)+Bu(t)+K[C\hat{x}(t_i)-y(t)] \quad (14)$$

when $t\in[t_i,t_{i+1})$ is an observer for the system (8). Moreover, the error variable $\tilde{x}=x-\hat{x}$, satisfies for all $t\in[t_i,t_{i+1})$ the following error equation, which is input-to-state stable (see [35]) with respect to (δ_1,δ_2),

$$\dot{\tilde{x}}(t) \;=\; H\tilde{x}(t)+L\left[\mathscr{M}(t,t_i)^{-1}-I\right]\tilde{x}(t)+\delta_4(t,t_i), \quad (15)$$

where

$$\mathscr{M}(t,s)=e^{A(t-s)}+\left(\int_s^t e^{A(t-\ell)}d\ell\right)L \quad (16)$$

and

$$\delta_4(t,s) \;=\; \delta_3(t)-L\mathscr{M}(t,s)^{-1}\int_s^t e^{A(t-\ell)}\delta_3(\ell)d\ell, \quad (17)$$

with

$$\delta_3(t)=\delta_1(t)+K\delta_2(t). \quad (18)$$

Discussion of Theorem 1.

- Assumption 1 is a detectability condition, which ensures that an observer for the system $\dot{x}(t)=Ax(t)$ with the output $Cx(t)$ can be constructed.
- When $A\neq 0$, the constant a_* can be chosen equal to $||A||$. Moreover, since $L\neq 0$ and S is invertible, it follows that $SL\neq 0$. Therefore the constants in Assumption 2 are well-defined and positive.
- In Assumption 1, we have assumed that $L\neq 0$. This simplifies the statements and proofs of our results, but is by no means necessary.
- The term Bu in (8) plays no direct role in the context of the construction of an observer of the type (14) for the system (8). However, its presence shows that our results apply in the context of systems whose inputs are used to give them desirable properties.

Proof. Since δ_1 and δ_2 are piecewise-continuous functions of t, the system (8)-(14) is forward-complete (see [20] for the definition of forward-complete systems). Next, we write the error equation. We obtain, for all $t\in[t_i,t_{i+1})$,

$$\dot{\tilde{x}}(t) \;=\; Ax(t)-A\hat{x}(t)-K[C\hat{x}(t_i)-y(t)]+\delta_1(t). \quad (19)$$

Since $L=KC$, (19) can be rewritten, for all $t\in[t_i,t_{i+1})$, as

$$\dot{\tilde{x}}(t) \;=\; A\tilde{x}(t)+L\tilde{x}(t_i)+\delta_3(t), \quad (20)$$

where δ_3 is the function defined in (18). By multiplying both sides of (20) by e^{-At} and integrating, one can prove that, for all $t\in[t_i,t_{i+1})$,

$$\mathscr{M}(t,t_i)\tilde{x}(t_i) \;=\; \tilde{x}(t)-\int_{t_i}^t e^{A(t-\ell)}\delta_3(\ell)d\ell, \quad (21)$$

where $\mathscr{M}$ is the function defined in (16). Observe that the requirement (12) in Assumption 2 and Lemma 2 in Appendix ensure that $\mathscr{M}(t,t_i)$ is invertible for all $t\in[t_i,t_{i+1})$. Thus, for all $t\in[t_i,t_{i+1})$,

$$\dot{\tilde{x}}(t) \;=\; \left[A+L\mathscr{M}(t,t_i)^{-1}\right]\tilde{x}(t)+\delta_4(t,t_i). \quad (22)$$

It follows that (15) is satisfied for all $t\in[t_i,t_{i+1})$.

Next, we analyze the stability of the system (15) through a Lyapunov approach. To conduct this analysis, we introduce the positive-definite quadratic function

$$\mathscr{S}(\xi)=\xi^\top S\xi, \quad (23)$$

where S is a symmetric positive-definite matrix such that (11) is satisfied. Its derivative along the trajectories of (15) satisfies, for all $t \in [t_i, t_{i+1})$,

$$\begin{array}{rcl} \dot{\mathscr{S}}(t) & = & 2\tilde{x}(t)^\top S\left[H + L(\mathscr{M}(t,t_i)^{-1} - I)\right]\tilde{x}(t) \\ & & +2\tilde{x}(t)^\top S\delta_4(t,t_i). \end{array} \tag{24}$$

By (11) and the triangle inequality, it follows that, for all $t \in [t_i, t_{i+1})$,

$$\begin{array}{rcl} \dot{\mathscr{S}}(t) & \leq & -\frac{3}{4}||\tilde{x}(t)||^2 \\ & & +2||SL||\left|\left|\mathscr{M}(t,t_i)^{-1} - I\right|\right|||\tilde{x}(t)||^2 \\ & & +4||S||^2||\delta_4(t,t_i)||^2. \end{array} \tag{25}$$

On the other hand, the requirement (12) in Assumption 2 and Lemma 2 imply that, for all $t \in [t_i, t_{i+1})$,

$$\left|\left|\mathscr{M}(t,t_i)^{-1} - I\right|\right| \leq \left(\frac{2||L||}{a_*} + 1\right)(e^{a_*\tau} - 1)\, e^{a_*\tau} \tag{26}$$

and therefore

$$\begin{array}{rcl} \dot{\mathscr{S}}(t) & \leq & -\frac{3}{4}||\tilde{x}(t)||^2 \\ & + & 2||SL||\left(\frac{2}{a_*} + 1\right)(e^{a_*\tau} - 1)\, e^{a_*\tau}||\tilde{x}(t)||^2 \\ & + & 4||S||^2||\delta_4(t,t_i)||^2. \end{array} \tag{27}$$

From the requirement (13) in Assumption 2, we deduce that, for all $t \in [t_i, t_{i+1})$,

$$\dot{\mathscr{S}}(t) \quad \leq \quad -\tfrac{1}{2}||\tilde{x}(t)||^2 + 4||S||^2||\delta_4(t,t_i)||^2. \tag{28}$$

Finally, through lengthy but simple calculation, one can prove that the ISS inequality

$$||\tilde{x}(t)|| \leq c_3 e^{\frac{s-t}{4||S||}}||\tilde{x}(s)|| + c_2 \sup_{\ell \in [s,t]} (||\delta_1(\ell)|| + ||\delta_2(\ell)||), \tag{29}$$

with $c_2 = \sqrt{\frac{2||S||c_2}{\lambda_S}}$, $c_3 = \sqrt{\frac{||S||}{\lambda_S}}$, is satisfied.

IV. Interval observer for systems with discrete-time measurements

In this section, our goal it to construct interval observers for the system (8) under Assumptions 1 and 2.

A. Preliminary step

Before constructing interval observers, we need to introduce a new assumption that pertains to the disturbances in (8) and establish technical results.

Assumption 3: A continuous function $\delta^d : [0,+\infty) \to [0,+\infty)$ is known, such that for all $t \geq 0$ and $l = 1,2$,

$$||\delta_l(t)|| \leq \delta^d(t). \tag{30}$$

We assume that the system (8) satisfies Assumptions 1 to 3. Then Theorem 1 applies and leads to the error equation (15). To facilitate the design of interval observers, we need to transform this error equation into an equation that is cooperative in the absence of δ_4 and of the term $L\left[\mathscr{M}(t,t_i)^{-1} - I\right]\tilde{x}(t)$. This can be done by applying the technique of [23]. Since H is a constant and Hurwitz matrix, [23] shows how to build a C^∞ function $\mathscr{P} : \mathbb{R} \to \mathbb{R}^{n\times n}$ that is invertible for all $t \in \mathbb{R}$, bounded in norm with a first derivative bounded in norm and such that, for all $t \in \mathbb{R}$,

$$\dot{\mathscr{P}}(t) = -\mathscr{P}(t)H + G\mathscr{P}(t), \tag{31}$$

where G is a constant cooperative and Hurwitz matrix and $\mathscr{T} : \mathbb{R} \to \mathbb{R}^{n\times n}$, $\mathscr{T}(t) = \mathscr{P}(t)^{-1}$ for all $t \in \mathbb{R}$ is a C^∞ function bounded in norm. It follows that there exist a symmetric and positive-definite matrix Q such that the matrix inequality

$$QG + G^\top Q \preccurlyeq -I \tag{32}$$

is satisfied and a positive real number

$$p_* = \sup_{t\in\mathbb{R}}\{||\mathscr{P}(t)||, ||\dot{\mathscr{P}}(t)||, ||\mathscr{P}(t)^{-1}||\}. \tag{33}$$

Now, we introduce the time-varying change of coordinates

$$m(t) = \mathscr{P}(t)\tilde{x}(t). \tag{34}$$

Using (15), elementary calculations give

$$\begin{array}{l} \dot{m}(t) = \dot{\mathscr{P}}(t)\tilde{x}(t) \\ \quad +\mathscr{P}(t)\left[H\tilde{x}(t) + L(\mathscr{M}(t,t_i)^{-1} - I)\tilde{x}(t) + \delta_4(t,t_i)\right]. \end{array} \tag{35}$$

From (31), it follows that

$$\begin{array}{rl} \dot{m}(t) = & G\mathscr{P}(t)\tilde{x}(t) + \mathscr{P}(t)L(\mathscr{M}(t,t_i)^{-1} - I)\tilde{x}(t) \\ & +\mathscr{P}(t)\delta_4(t,t_i) \\ = & Gm(t) \\ & +\mathscr{P}(t)L(\mathscr{M}(t,t_i)^{-1} - I)\mathscr{P}(t)^{-1}m(t) \\ & +\mathscr{P}(t)\delta_4(t,t_i). \end{array} \tag{36}$$

This leads us to the system

$$\dot{m}(t) = [G + \mathscr{R}(t,t_i)]m(t) + \delta_5(t,t_i), \tag{37}$$

with

$$\begin{array}{l} \mathscr{R}(t,t_i) = \mathscr{P}(t)L\left[\mathscr{M}(t,t_i)^{-1} - I\right]\mathscr{P}(t)^{-1}, \\ \delta_5(t,t_i) = \mathscr{P}(t)\delta_4(t,t_i). \end{array} \tag{38}$$

It is worth noticing that the system $\dot{m}(t) = Gm(t)$ is cooperative but not necessarily the system $\dot{m}(t) = [G + \mathscr{R}(t,t_i)]m(t)$.

One can check easily that the definition of δ_4 in (17) and (57) in Lemma 2 imply that, for all $t \in [t_i, t_{i+1})$,

$$||\delta_4(t,t_i)|| \leq ||\delta_3(t)|| + 2||L||e^{2a_*\tau}\int_{t_i}^t ||\delta_3(\ell)||d\ell. \tag{39}$$

This inequality, Assumption 2, the definition of p_* in (33) and $\delta_3 = \delta_1 + K\delta_2$ imply that

$$\begin{array}{l} ||\mathscr{P}(t)\delta_4(t,t_i)|| \leq p_*||\delta_1(t) + K\delta_2(t)|| \\ \qquad + 2p_*||L||e^{2a_*\tau}\int_{t_i}^t ||\delta_1(\ell) + K\delta_2(\ell)||d\ell \end{array} \tag{40}$$

for all $t \in [t_i, t_{i+1})$. From Assumption 3 we deduce that, for all integer $i \in \mathbb{N}$ and for all $t \in [t_i, t_{i+1})$, the inequalities

$$-\delta_*(t) \leq \delta_5(t,t_i) \leq \delta_*(t) \tag{41}$$

with

$$\delta_*(t) = p_*(1 + ||K||)\psi(t)(1 \ldots 1)^\top, \tag{42}$$

with $\psi(t) = ||\delta^d(t)|| + 2||L||e^{2a_*\tau}\int_{\max\{0,t-\tau\}}^t ||\delta^d(\ell)||d\ell$, are satisfied. Notice that the function δ_* is continuous.

B. Interval observer

Denote the entries of the matrix function $\mathscr{R}(t,t_i)$ defined in (38) by $r_{k,l}(t,t_i)$ and the entries of the matrix G by $g_{k,l}$ and recall that $\mathscr{T}^{-1}(t)=\mathscr{P}(t)$. Let

$$\begin{array}{c}\mathscr{R}^+(t,t_i)=(r^+_{k,l}(t,t_i)),\\ \mathscr{R}^-(t,t_i)=\mathscr{R}^+(t,t_i)-\mathscr{R}(t,t_i),\\ \mathscr{P}^+(t)=\max\{\mathscr{P}(t),0\},\\ \mathscr{P}^-(t)=\mathscr{P}^+(t)-\mathscr{P}(t),\\ \mathscr{T}^+(t)=\max\{\mathscr{P}(t)^{-1},0\}\\ \mathscr{T}^-(t)=\mathscr{T}^+(t)-\mathscr{P}(t)^{-1},\end{array}\tag{43}$$

with $r^+_{k,l}(t,t_i)=r_{k,l}(t,t_i)$ if $k=l$ or $g_{k,l}+r_{k,l}(t,t_i)\geq 0$ and $r^+_{k,l}(t,t_i)=0$ if $k\neq l$ and $g_{k,l}+r_{k,l}(t,t_i)\leq 0$.

Observe for later use that $G+\mathscr{R}^+$ is a cooperative function and all functions $\mathscr{R}^-$, $\mathscr{P}^+$, $\mathscr{P}^-$, $\mathscr{T}^+$ and $\mathscr{T}^-$ are nonnegative.

We are ready to prove the following result.

Theorem 2: Consider the system (8) under Assumptions 1 to 3. Let G and p_* be the matrix and the constant defined in Section IV-A. Then the system described by

$$\left\{\begin{array}{rcl}\dot{\hat{x}}(t)&=&A\hat{x}(t)+Bu(t)+K[C\hat{x}(t_i)-y(t)],\\ \dot{m}^+(t)&=&[G+\mathscr{R}^+(t,t_i)]m^+(t)\\ &&-\mathscr{R}^-(t,t_i)m^-(t)+\delta_*(t),\\ \dot{m}^-(t)&=&[G+\mathscr{R}^+(t,t_i)]m^-(t)\\ &&-\mathscr{R}^-(t,t_i)m^+(t)-\delta_*(t),\end{array}\right.\tag{44}$$

when $t\in[t_i,t_{i+1})$, with δ_* defined in (42), and associated at $t=s_0\geq 0$ with the initial conditions

$$\begin{pmatrix}\hat{x}_0\\ m_0^+\\ m_0^-\end{pmatrix}=\begin{pmatrix}\hat{x}_0\\ \mathscr{P}^+(s_0)\tilde{x}_0^+-\mathscr{P}^-(s_0)\tilde{x}_0^-\\ \mathscr{P}^+(s_0)\tilde{x}_0^--\mathscr{P}^-(s_0)\tilde{x}_0^+\end{pmatrix}\tag{45}$$

with $\tilde{x}_0^+=x_0^+-\hat{x}_0$, $\tilde{x}_0^-=x_0^--\hat{x}_0$ and the bounds

$$\begin{array}{rcl}x^+(t)&=&\hat{x}(t)+\mathscr{T}^+(t)m^+(t)-\mathscr{T}^-(t)m^-(t),\\ x^-(t)&=&\hat{x}(t)+\mathscr{T}^+(t)m^-(t)-\mathscr{T}^-(t)m^+(t),\end{array}\tag{46}$$

is an interval observer for the system (8) when either for all integer $i\in\mathbb{N}$ and all $t\in[t_i,t_{i+1})$, the matrix $G+\mathscr{R}^+(t,t_i)$ is cooperative or

$$\tau\in(0,\tau_B]\,,\tag{47}$$

with

$$\tau_B=\tfrac{1}{a_*}\ln\left(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+b_*}\right),\tag{48}$$

with

$$b_*=\tfrac{2a_*}{\left(||SL||+2p_*^4\sqrt{n}||S||||L||\right)(2||L||+a_*)}.$$

Proof. Since the functions $\rho^+:\mathbb{R}\to\mathbb{R}^{n\times n}$ and $\rho^-:\mathbb{R}\to\mathbb{R}^{n\times n}$ defined by $\rho^+(t)=\mathscr{R}^+(t,t_i)$, $\rho^-(t)=\mathscr{R}^-(t,t_i)$ for all $t\in[t_i,t_{i+1})$ and all positive integers i are discontinuous, the (m^+,m^-)-subsystem (44) is discontinuous with respect to t. However ρ^+ and ρ^- are piecewise-continuous. Hence, existence and uniqueness of the solutions are guaranteed. The solutions of the systems (8) and (44) are thus defined over $[0,+\infty)$ when $u(\cdot)$, $\delta_*(\cdot)$ and $\delta_1(\cdot)$ are bounded on any interval $[0,t)$, $t>0$.

Now, for the sake of simplicity, we consider the initial time $s_0=0$. The case where $s_0>0$ can be handled similarly.

Let $x_0\in\mathbb{R}^n$ be an initial condition of (8) at the instant $t=0$. Let $x_0^+\in\mathbb{R}^n$, $x_0^-\in\mathbb{R}^n$ be such that $x_0^-\leq x_0\leq x_0^+$. Let $(\hat{x}_0,m_0^+,m_0^-)\in\mathbb{R}^{3n}$, be an initial condition of (44) at the instant $t=0$ satisfying

$$\begin{array}{l}m_0^+=\mathscr{P}^+(0)(x_0^+-\hat{x}_0)-\mathscr{P}^-(0)(x_0^--\hat{x}_0),\\ m_0^-=\mathscr{P}^+(0)(x_0^--\hat{x}_0)-\mathscr{P}^-(0)(x_0^+-\hat{x}_0).\end{array}\tag{49}$$

Next, we consider the solutions of (8) and (44) with respectively x_0 and $(\hat{x}_0,m_0^+,m_0^-)$ as initial condition at $t=0$. We denote these solutions $(x(t),\hat{x}(t),m^+(t),m^-(t))$. Since the functions $\mathscr{P}^+$ and $\mathscr{P}^-$ are nonnegative, the inequalities

$$\begin{array}{l}\mathscr{P}^+(0)x_0^-\leq\mathscr{P}^+(0)x_0\leq\mathscr{P}^+(0)x_0^+,\\ \mathscr{P}^-(0)x_0^-\leq\mathscr{P}^-(0)x_0\leq\mathscr{P}^-(0)x_0^+,\end{array}\tag{50}$$

are satisfied. It follows that

$$\begin{array}{l}\mathscr{P}^+(0)x_0^--\mathscr{P}^-(0)x_0^+\leq\mathscr{P}(0)x_0,\\ \mathscr{P}(0)x_0\leq\mathscr{P}^+(0)x_0^+-\mathscr{P}^-(0)x_0^-.\end{array}\tag{51}$$

From the equality $\mathscr{P}^+(0)-\mathscr{P}^-(0)=\mathscr{P}(0)$, (49) and (51), we deduce that $m_0^-\leq\mathscr{P}(0)x_0-\mathscr{P}(0)\hat{x}_0\leq m_0^+$, or, equivalently,

$$m_0^-\leq\mathscr{P}(0)\tilde{x}_0\leq m_0^+,\tag{52}$$

with $\tilde{x}_0=x_0-\hat{x}_0$. On the other hand, we know that $m(t)=\mathscr{P}(t)\tilde{x}(t)$, with $\tilde{x}(t)=x(t)-\hat{x}(t)$ and $m_0=\mathscr{P}(0)\tilde{x}_0$ as initial condition, is solution of (37). Our next objective is to prove that, for all $t\geq 0$, $m^-(t)\leq m(t)\leq m^+(t)$, where $m^+(t)$ and $m^-(t)$ are the components of the solution defined above. To prove this, we introduce the notation

$$\overline{m}^+(t)=m^+(t)-m(t),\ \overline{m}^-(t)=m(t)-m^-(t).\tag{53}$$

Bearing (37) and (44) in mind, using (43) and grouping the terms, we obtain, for all $t\in[t_i,t_{i+1})$,

$$\left\{\begin{array}{rcl}\dot{\overline{m}}^+(t)&=&[G+\mathscr{R}^+(t,t_i)]\overline{m}^+(t)\\ &&+\mathscr{R}^-(t,t_i)\overline{m}^-(t)+\delta_*(t)-\delta_5(t,t_i),\\ \dot{\overline{m}}^-(t)&=&[G+\mathscr{R}^+(t,t_i)]\overline{m}^-(t)\\ &&+\mathscr{R}^-(t,t_i)\overline{m}^+(t)+\delta_5(t,t_i)+\delta_*(t).\end{array}\right.\tag{54}$$

From (52), we deduce that $\overline{m}^+(0)\geq 0$ and $\overline{m}^-(0)\geq 0$. Since $G+\mathscr{R}^+(t,t_i)$ is cooperative for all $i\in\mathbb{N}$ and $t\in[t_i,t_{i+1})$ and the functions $\mathscr{R}^-(t,t_i)$, $\delta_*(t)-\delta_5(t,t_i)$, $\delta_5(t,t_i)+\delta_*(t)$ are nonnegative (see (43) and (41)), it follows that $\overline{m}^+(t)\geq 0$ and $\overline{m}^-(t)\geq 0$ for all $t\in[t_0,t_1)$. Since the solution $(\overline{m}^+(t),\overline{m}^-(t))$ is continuous, it follows that $\overline{m}^+(t_1)\geq 0$ and $\overline{m}^-(t_1)\geq 0$. Next, we prove by induction that, for any integer $i\in\mathbb{N}$, and any $t\in[t_i,t_{i+1}]$, $\overline{m}^+(t)\geq 0$ and $\overline{m}^-(t)\geq 0$. Consequently, the inequalities $m^-(t)\leq m(t)\leq m^+(t)$ are satisfied for all $t\geq 0$. From this inequality, we can deduce, through calculations omitted for the sake of conciseness, that for all $t\geq 0$, $x^-(t)\leq x(t)\leq x^+(t)$, with $(x^+(t),x^-(t))$ defined in (46). Consequently, (44) with the initial conditions (45) and the bounds (46), is a framer for (8). Thus, to prove that (44) is an interval observer for the system (8), it remains to demonstrate that $\lim\limits_{t\to+\infty}||x^+(t)-x^-(t)||=0$ when δ_* is not present. Since the functions $||\mathscr{T}^+||$ and $||\mathscr{T}^-||$ are bounded, the equality $\lim\limits_{t\to+\infty}||x^+(t)-x^-(t)||=0$ is satisfied if $\lim\limits_{t\to+\infty}||m^+(t)-m^-(t))||=0$. Due to space limitation, the proof of this is omitted.

V. Conclusion

Under a detectability assumption, we have constructed a family of interval observers for all linear time-invariant systems with bounded additive disturbances and discrete-time measurements affected by bounded additive noise. Much remains to be done. A comparison to an approach where the discrete-time measurements are taken into account by set-inversion via interval analysis [16], [17] will be the subject of a future study. Extensions to nonlinear, time-varying systems or to systems with delay may be also the subject of future work.

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Appendix

Lemma 1: Let $M \in \mathbb{R}^{l\times l}$ be a matrix whose entries are denoted by m_{ij}. Let $N \in \mathbb{R}^{l\times l}$ be any matrix, whose entries n_{ij} are such that either $n_{ij} = m_{ij}$ or $n_{ij} = 0$. Then $||N|| \leq \sqrt{n}||M||$.
Proof. The proof is omitted because of page limitation.

Lemma 2: Let $A \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{n\times n}$ be constant matrices. Let $\mathscr{M}$ be the function defined in (16). Let τ be a real number such that

$$0 < \tau \leq \frac{1}{a_*} \ln\left(1 + \frac{a_*}{2||L||}\right) \tag{55}$$

where a_* is a real number such that $a_* > 0$, $a_* \geq ||A||$.

Then for all $s \in \mathbb{R}$ and $t \in [s, s+\tau]$, the matrix $\mathscr{M}(t,s)$ is invertible and, for all $s \in \mathbb{R}$ and $t \in [s, s+\tau]$

$$||\mathscr{M}(t,s)^{-1} - I|| \leq \left(2\frac{||L||}{a_*} + 1\right)(e^{a_*\tau} - 1)\, e^{a_*\tau} \tag{56}$$

and

$$||\mathscr{M}(t,s)^{-1}|| \leq 2e^{a_*\tau}. \tag{57}$$

Proof. The proof is omitted because of page limitation.