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Equidimensional triangularization of multidimensional linear systems

Alban Quadrat

Abstract—Based on the results obtained in [12] on the purity filtration of a finitely presented module associated with a multidimensional linear system, this paper aims at obtaining an equivalent block-triangular representation of the multidimensional linear system defined by equidimensional diagonal blocks. The multidimensional linear system can then be integrated in cascade by solving equidimensional homogeneous linear systems. Many multidimensional linear systems defined by under/overdetermined linear systems of partial differential equations can be explicitly solved by means of the PurityFiltration and AbelianSystems packages, but cannot be computed by classical computer algebra systems such as Maple. The results developed in this paper generalize those obtained in the literature on Monge parametrizations and on the classification of autonomous elements by their codimensions.

I. INTRODUCTION

This paper is the continuation of [12]. We refer the reader to [12] for the notations and the results used in what follows.

We recall that $D$ denotes a noetherian domain with a finite global dimension $\text{gl}(D) = n$ and which satisfies

$$\forall i \in \mathbb{N} = \{0, 1, 2, \ldots\}, \quad \text{ext} D^i_\text{D}(\text{ext}^i_\text{D}(M, D), D) = 0,$$

for all left $D$-modules $M$. For more details, see [5], [12]. In particular, these conditions hold for a Auslander regular ring $D$ [4], [5] such as, for instance, the commutative polynomial ring $D = k[x_1, \ldots, x_n]$ in $x_1, \ldots, x_n$ with coefficients in a field $k$ or the noncommutative polynomial rings $A_n(k)$ (resp., $B_n(k), D_n(k)$ and $D_n(k)$) of partial differential (PD) operators in $\partial_i = \frac{\partial}{\partial x_i}, i = 1, \ldots, n$, with coefficients in the ring $k[x_1, \ldots, x_n]$ (resp., the ring $k[x_1, \ldots, x_n]$ of rational functions, the ring $k[x_1, \ldots, x_n]$ of formal power series, or the ring $\mathbb{R}\{x_1, \ldots, x_n\}$ of locally convergent power series), where $k$ is a field of characteristic 0 (e.g., $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$) [4].

The purpose of this paper is to apply Theorem 3.1 and Corollary 3.1 of [12] to the short exact sequences defined in (42) of [12] to determine a new finite presentation of the left $D$-module $M = D^1 x / (D^1 x q R)$ defined by a block-triangular matrix $P$ formed by block-diagonal matrices defining finite presentations of the pure left $D$-modules $M / t(M), \ker \gamma_21$ and $\ker \gamma_32$ defined in [12], and of the left $D$-module $\text{ext}^1_\text{D}(N_{33}, D)$ of grade greater or equal to 3 (which is also pure when $\ker D(R_3) = 0$). For more details and definitions, we refer the reader to [12].

II. MAIN RESULTS

Let us now precisely describe the left $D$-homomorphisms $\gamma_32$ and $\gamma_21$ and the left $D$-modules $\ker \gamma_32$ and $\ker \gamma_21$ defined in [12] (see (41) and (42)). Applying the contravariant left exact functor $\text{hom}_D(\cdot, D)$ to the commutative diagram defined in Fig. 3 of [12], we obtain the following commutative diagram formed by horizontal complexes:

$$
\begin{align*}
D^1 x_{-13} & \xrightarrow{R_{03}} D^1 x_{p03} & \xrightarrow{R_{13}} D^1 x_{p13} \\
D^1 x_{-12} & \xrightarrow{R_{02}} D^1 x_{p02} & \xrightarrow{R_{12}} D^1 x_{p12} \\
D^1 x_{-11} & \xrightarrow{R_{01}} D^1 x_{p01} & \xrightarrow{R_{11}} D^1 x_{p11}.
\end{align*}
$$

(1)

Using (34) of [12], the defect of exactness of the top (resp., middle, bottom) horizontal complex of (1) is $\text{ext}^1_D(N_{13}, D)$ (resp., $\text{ext}^2_D(N_{12}, D), \text{ext}^1_D(N_{11}, D)$). Let us introduce the canonical projections defined in (2). The commutative diagram (1) induces the two left $D$-homomorphisms:

$$
\begin{align*}
\ker D(R_{03})/(D^1 x_{p13} R_{13}) & \xrightarrow{\alpha_{32}} \ker D(R_{02})/(D^1 x_{p12} R_{12}) \\
\rho_3(\lambda) & \xrightarrow{\rho_2(\mu)} \rho_2(\lambda F_{03}).
\end{align*}
$$

(3)

Two chases in the commutative diagram (1) show that $\rho_3$ and $\rho_2$ are well-defined (see, e.g., [17]).

Let us now use Proposition 2.2 of [12] to get a finite presentation of the left $D$-modules $\text{ext}^1_D(N_{33}, D), \text{ext}^2_D(N_{32}, D)$ and $\text{ext}^1_D(N_{11}, D)$. Let $R_{1k} \in D^1 x_{p1k}$ be such that $\ker D(R_{0k}) = D^1 x_{p1k} R_{1k}$ for $k = 1, 2, 3$. Since $D^1 x_{p1k} R_{1k} \subseteq D^1 x_{p1k} R_{1k}^\prime$, there exists $R_{1k}'' \in D^1 x_{p1k}$ such that:

$$R_{1k}'' = R_{1k} R_{1k}^\prime.
$$

(5)

Let $R_{2k}'' \in D^1 x_{p2k}$ be a matrix such that $\ker D(R_{2k}''') = D^1 x_{p2k} R_{2k}$, then, using Proposition 2.2 of [12], we obtain the left $D$-homomorphism defined by (6), where

$$L_k \triangleq D^1 x_{p1k} / (D^1 x_{p1k} R_{1k}'' + D^1 x_{p2k} R_{2k}''),$$

and $\rho_k' : D^1 x_{p1k} \rightarrow L_k$ is the canonical projection.

Since $R_{1k}'' F_{0k} R_{0(k-1)} = R_{1k}'' R_{0k} F_{-1k} = 0$, then $D^1 x_{p1k} (R_{1k}'' F_{0k}) \subseteq \ker D(R_{0(k-1)}) = D^1 x_{p1k} R_{1(k-1)}$, and thus there exists $F_{1k}'' \in D^1 x_{p1k}(k-1)$ such that:

$$\forall k = 2, 3, \quad R_{1k}'' F_{0k} = F_{1k}'' R_{1(k-1)}.'
$$

(7)
\[
\begin{align*}
\rho_3 : \ker_D(R_{03}) & \longrightarrow \ker_D(R_{03})/(D^{1 \times p_{13}} R_{13}) \cong \text{ext}^1_D(N_{13}, D) \cong \text{ext}^3_D(N_{33}, D), \\
\rho_2 : \ker_D(R_{02}) & \longrightarrow \ker_D(R_{02})/(D^{1 \times p_{12}} R_{12}) \cong \text{ext}^1_D(N_{12}, D) \cong \text{ext}^2_D(N_{22}, D), \\
\rho_1 : \ker_D(R_{01}) & \longrightarrow \ker_D(R_{01})/(D^{1 \times p_{11}} R_{11}) \cong \text{ext}^1_D(N_{11}, D) \cong t(M).
\end{align*}
\]  

(2)

\[
\chi_k : L_k \cong D^{1 \times p_{k}}/(D^{1 \times p_{k}} R_{1k}^\prime + D^{1 \times p_{2k}} R_{2k}^\prime) \longrightarrow (D^{1 \times p_{k}} R_{1k}^\prime)/(D^{1 \times p_{k}} R_{1k}) \cong \text{ext}^1_D(N_{1k}, D), \\
\rho_k^\prime(\lambda) & \longrightarrow \rho_k(\lambda R_{1k}).
\]

(6)

Similarly, there exists \( F_{2k}^\prime \in D^{p_{2k} \times p_{2(k-1)}} \) such that:

\[
\forall k = 2, 3, \quad R_{2k}^\prime F_{1k}^\prime = F_{2k}^\prime R_{a(k-1)}^\prime.
\]

(8)

Thus, we obtain the commutative exact diagram (9).

**Remark 2.1:** If \( R_{0k} = 0 \), i.e., \( \ker_D(R_{1k}) = 0 \), then applying the functor \( \text{Hom}_D(\cdot, D) \) to the short exact sequence \( 0 \longrightarrow D^{p_{1k}} \xrightarrow{R_{1k}} D^{p_{1k} \times p_{1k}} N_{1k} \longrightarrow 0 \), we get the complex \( 0 \longrightarrow D^{1 \times p_{1k}} \xrightarrow{R_{1k}} D^{1 \times p_{1k}} \), which yields \( \ker_D(R_{0k}) = D^{1 \times p_{1k}} \), i.e.:

\[
R_{1k}^\prime = I_{p_{1k}}, \quad p_{1k} = p_{0k}, \quad R_{2k}^\prime = 0.
\]

Let us now deduce two identities which will be used in what follows. Combining (30) of [12] for \( k = 2 \) with (5) for \( k = 1 \) and \( k = 2 \), and with (7) for \( k = 2 \), we obtain

\[
R_{11}^\prime R_{11}^\prime = R_{11} = R_{12} F_{02} = R_{12} R_{12} F_{02} = R_{12}^\prime R_{12}^\prime R_{11}^\prime,
\]

and thus \( R_{11}^\prime - R_{12}^\prime R_{12}^\prime R_{11}^\prime = 0 \), i.e.,

\[
D^{1 \times p_{11}} (R_{11}^\prime - R_{12}^\prime R_{12}^\prime) \subseteq \ker_D(R_{11}^\prime) = D^{1 \times p_{11}} R_{21}^\prime,
\]

which proves the existence of \( X_{12} \in D^{p_{11} \times p_{21}} \) such that:

\[
R_{11}^\prime = R_{12}^\prime F_{12}^\prime + X_{12} R_{21}^\prime.
\]

(10)

Combining (31) of [12] for \( k = 3 \) with (5) for \( k = 2 \) and \( k = 3 \), and with (7) for \( k = 2 \), we obtain

\[
F_{13} (R_{12}^\prime R_{12}^\prime) = F_{13} R_{12} = R_{13} F_{03} = (R_{13}^\prime R_{13}^\prime) F_{03} = R_{13}^\prime F_{13} R_{13}^\prime R_{12}^\prime,
\]

and thus \( F_{13} (R_{12}^\prime R_{12}^\prime - R_{13}^\prime F_{13} R_{13}^\prime R_{12}^\prime) = 0 \), i.e.,

\[
D^{1 \times p_{13}} (F_{13} R_{12}^\prime R_{12}^\prime - R_{13}^\prime F_{13} R_{13}^\prime R_{12}^\prime) \subseteq \ker_D(R_{12}^\prime) = D^{1 \times p_{12}} R_{22}^\prime,
\]

which proves the existence of \( X_{22} \in D^{p_{13} \times p_{22}} \) such that:

\[
F_{13} R_{12}^\prime - R_{13}^\prime F_{13} R_{13}^\prime X_{22} R_{22}^\prime.
\]

(11)

Using \( t(M) \cong \text{ext}^1_D(N_{12}, D) \) (see (2) and [12]) and the isomorphisms \( \chi_k \)'s defined by (6), we get:

\[
\left\{
\begin{array}{l}
L_1 = D^{1 \times p_{11}}/(D^{1 \times p_{11}} R_{11}^\prime + D^{1 \times p_{21}} R_{21}^\prime) \\
\cong \text{ext}^1_D(N_{11}, D) \cong t(M), \\
L_2 = D^{1 \times p_{12}}/(D^{1 \times p_{12}} R_{12}^\prime + D^{1 \times p_{22}} R_{22}^\prime) \\
\cong \text{ext}^2_D(N_{22}, D), \\
L_3 = D^{1 \times p_{13}}/(D^{1 \times p_{13}} R_{13}^\prime + D^{1 \times p_{23}} R_{23}^\prime) \\
\cong \text{ext}^3_D(N_{33}, D).
\end{array}
\right.
\]

(12)

Then, we can define the left \( D \)-homomorphism

\[
\bar{\chi}_{32} = \chi_2^{-1} \circ \alpha_{32} \circ \chi_3 : L_3 \longrightarrow L_2,
\]

where the \( \chi_i \)'s are defined by (6) and \( \alpha_{32} \) is defined by (3). Using (7) for \( k = 3 \), we have

\[
\bar{\alpha}_{32}(\rho_3(\lambda)) = (\chi_2^{-1} \circ \alpha_{32})(\rho_3(\lambda R_{13}^\prime F_{03})) = \chi_2^{-1} \circ \rho_2(\lambda R_{13}^\prime F_{03}) = \rho_2(\lambda F_{13}^\prime),
\]

for all \( \lambda \in D^{1 \times p_{13}}. \) Using (11) and (8) for \( k = 3 \), we get

\[
\begin{pmatrix}
R_{13}^\prime \\
R_{23}^\prime
\end{pmatrix}
\begin{pmatrix}
F_{13} \\
F_{23}
\end{pmatrix} = \begin{pmatrix}
F_{13} R_{12} - X_{22} R_{22}^\prime \\
F_{23} R_{22}^\prime
\end{pmatrix},
\]

which yields the commutative exact diagram (14).

Up to isomorphism, the short exact sequence

\[
0 \longrightarrow \text{ext}^3_D(N_{33}, D) \xrightarrow{\gamma_{32}} \text{ext}^2_D(N_{22}, D) \longrightarrow \text{coker} \gamma_{32} \longrightarrow 0
\]

(see [12]) becomes the following short exact sequence:

\[
0 \longrightarrow L_3 \xrightarrow{\bar{\alpha}_{32}} L_2 \xrightarrow{\theta_2} \text{coker} \gamma_{32} \longrightarrow 0.
\]

(15)

Using 3 of Proposition 3.1 of [8], the left \( D \)-module \( \text{coker} \bar{\alpha}_{32} \) is then defined by:

\[
\text{coker} \bar{\alpha}_{32} = D^{1 \times p_{12}}/(D^{1 \times p_{13}} F_{13}^\prime + D^{1 \times p_{12}} R_{12}^\prime + D^{1 \times p_{22}} R_{22}^\prime).
\]

Then, we can easily check that the commutative exact diagram (16) holds, where \( \psi_2 : D^{1 \times (p_{13} + p_{12} + p_{12})} \longrightarrow L_3 \) is the left \( D \)-homomorphism defined by:

\[
\psi_2(e_i) = \begin{cases}
\rho_3(e_i) & i = 1, \ldots, p_{13}, \\
0 & i = p_{13} + 1, \ldots, p_{13} + p_{12} + p_{12}.
\end{cases}
\]

Applying Theorem 3.1 of [12] to the short exact sequence (15) with the matrix

\[
A = \begin{pmatrix}
I_{p_{13}}^\prime & 0 \\
0 & 0
\end{pmatrix} \in D^{p_{13} + p_{12} + p_{22} \times p_{13}}
\]

(see also Corollary 3.1 of [12]), we obtain the following characterization of the left \( D \)-module \( L_2 \) in terms of the presentations of \( L_3 \cong \text{ext}^3_D(N_{33}, D) \) and \( \text{coker} \bar{\alpha}_{32} \).
7
More precisely, using (Malgrange’s theorem (see Theorem 1.1 of [12]), we obtain:

If and let us consider the following left

\[ D_1 \times p_{13} \xrightarrow{\cdot R_{13}} D_1 \times p_{13} \quad D_1 \times p_{13} \xrightarrow{\cdot R'_{13}} D_1 \times p_{13} \quad D_1 \times p_{13} \xrightarrow{\cdot R_{23}} D_1 \times p_{13} \]

\[ D_1 \times p_{12} \xrightarrow{\cdot R_{12}} D_1 \times p_{12} \quad D_1 \times p_{12} \xrightarrow{\cdot R'_{12}} D_1 \times p_{12} \quad D_1 \times p_{12} \xrightarrow{\cdot R_{22}} D_1 \times p_{12} \]

\[ D_1 \times p_{11} \xrightarrow{\cdot R_{11}} D_1 \times p_{11} \quad D_1 \times p_{11} \xrightarrow{\cdot R'_{11}} D_1 \times p_{11} \quad D_1 \times p_{11} \xrightarrow{\cdot R_{21}} D_1 \times p_{11}. \]

\[ D_1 \times (p_{13} + p_{12} + p_{12}') \xrightarrow{\cdot R'_{13} + R'_{12} + R'_{22}} D_1 \times (p_{13} + p_{12} + p_{12}') \]

\[ (R_{13}' T R_{12}' T R_{22}' T)^T \]

\[ \downarrow \psi_2 \]

\[ 0 \rightarrow L_3 \xrightarrow{\pi_3} \]

\[ \rho_3 \]

\[ L_3 \rightarrow 0 \]

\[ \downarrow \]

\[ D_1 \times p_{12} \]

\[ R_{12}' + \rho_3 \]

\[ L_2 \rightarrow 0. \]

\[ \downarrow \]

\[ 0 \]

\[ \downarrow \]

\[ 0 \rightarrow L_3 \]

\[ \pi_3 \]

Corollary 2.1: If \( \mathcal{F} \) is a left \( D \)-module, then

\[ \ker_{\mathcal{F}}(Q_2) \cong \ker_{\mathcal{F}}(P_2), \]

and the following equivalence

\[ \begin{cases} R_{12}' v = 0, \\ R_{22}' v = 0, \end{cases} \]

holds under the following invertible transformations:

\[ \delta : \ker_{\mathcal{F}}(P_2) \longrightarrow \ker_{\mathcal{F}}(Q_2) \]

\[ \begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 12 \end{pmatrix} \]

\[ \begin{pmatrix} I_{12}^T F_{13}^T \end{pmatrix} v. \]
Moreover, using (10) and (8) for \( k = 2 \), we have

\[
\begin{pmatrix}
R''_{12} \\
R''_{22}
\end{pmatrix}F_{12}' = \begin{pmatrix}
R_{11}' - X_{12}R'_{21} \\
F_{22}'R_{21}'
\end{pmatrix}
= \begin{pmatrix}
I_{p_{11}} - X_{12} \\
0
\end{pmatrix} \begin{pmatrix}
F_{12}' \\
F_{22}'
\end{pmatrix},
\]

which yields the commutative exact diagram (20). Up to isomorphism, the short exact sequence

\[
0 \longrightarrow \text{ext}^2_{N_{22}}(N_{22}, D) \xrightarrow{\gamma_{21}} t(M) \longrightarrow \text{coker} \gamma_{21} \longrightarrow 0,
\]

(see [12]) becomes the following short exact sequence

\[
0 \longrightarrow L_2 \xrightarrow{\overline{\pi}_{21} \circ \varphi_{21}^{-1}} L_1 \xrightarrow{\vartheta_1} \text{coker} \overline{\pi}_{21} \longrightarrow 0, \quad (21)
\]

where, using 3 of Proposition 3.1 of [8], the left \( D \)-module \( \text{coker} \overline{\pi}_{21} \) is defined by:

\[
\text{coker} \overline{\pi}_{21} = D^{1 \times \nu_1}/(D^{1 \times \nu_2} F_{12}' + D^{1 \times \nu_1} R''_{11} + D^{1 \times \nu_2} R''_{21}).
\]

Using the left \( D \)-isomorphism \( \phi_{2}^{-1} : E_2 \longrightarrow L_2 \) defined by (17), the short exact sequence (21) yields the following short exact sequence

\[
0 \longrightarrow E_2 \xrightarrow{\overline{\pi}_{21} \circ \varphi_{21}^{-1}} L_1 \xrightarrow{\vartheta_1} \text{coker} \overline{\pi}_{21} \longrightarrow 0, \quad (22)
\]

where \( \overline{\pi}_{21} \circ \varphi_{21}^{-1} : E_2 \longrightarrow L_1 \) is defined by:

\[
(\overline{\pi}_{21} \circ \varphi_{21}^{-1})(\varphi_2(\nu)) = \overline{\pi}_{21} \left( \rho_{2}' \left( \nu \left( \frac{I_{p_{12}'} F_{13}'}{F_{12}' F_{13}'} \right) \right) \right) = \rho_{1}' \left( \nu \left( \frac{F_{12}'}{F_{13}'} \right) \right).
\]

Now, we can check that the commutative exact diagram (23) holds, where \( \psi_1 : D^{1 \times (p_{12}'+p_{11}+p_{21})} \longrightarrow E_2 \) is the left \( D \)-homomorphism defined by

\[
\psi_1(f_j) = \begin{cases} 
\varphi_2(f_j F), & j = 1, \ldots, p_{12}', \\
0, & j = p_{12}' + 1, \ldots, p_{21}' + p_{11} + p_{21},
\end{cases}
\]

where \( \{f_j\}_{j=1, \ldots, p_{12}'+p_{11}+p_{21}} \) is the standard basis of \( D^{1 \times (p_{12}'+p_{11}+p_{21})} \) and:

\[
F = \begin{pmatrix}
I_{p_{12}'} \\
0 \\
0 \\
0
\end{pmatrix} \in D^{(p_{12}'+p_{11}+p_{21}) \times (p_{12}'+p_{13}+p_{21})}.
\]

Applying Theorem 3.1 of [12] to the short exact sequence (22) with the matrix \( A = F \) (see Corollary 3.1 of [12]), we obtain the following proposition.

**Proposition 2.2:** With the hypotheses of Proposition 2.1 and the previous notations, let

\[
\rho' = p_{12}' + p_{11} + p_{21}' + p_{13}' + p_{12} + p_{22} + p_{13} + p_{23}',
\]

and let us consider the following two matrices

\[
P_1 = \begin{pmatrix}
F_{12}' & -I_{p_{12}'} \\
R_{11}' & 0 \\
R_{21}' & 0 \\
0 & F_{13}' & -I_{p_{13}'} \\
0 & R_{11}' & 0 \\
0 & R_{22}' & 0 \\
0 & 0 & R_{13}' \\
0 & 0 & R_{23}'
\end{pmatrix} \in D^{r \times (p_{12}'+p_{13}'+p_{21})},
\]

and the following two finitely presented left \( D \)-modules:

\[
\begin{align*}
P_1 &= D^{1 \times \rho_1'(\nu)/(D^{1 \times (p_{12}'+p_{13}'+p_{21})} Q_1),} \\
P_2 &= D^{1 \times (p_{12}'+p_{21}')} / (D^{1 \times \rho_1'(\nu)} P_1).
\end{align*}
\]

If \( \vartheta : L_1 \longrightarrow t(M) \) and \( \vartheta^{-1} : t(M) \longrightarrow L_1 \)

\[
\vartheta_1(\lambda) \longrightarrow \rho_1'(\lambda) \left( \begin{pmatrix}
I_{p_{12}'} \\
F_{12}' \\
F_{13}' \end{pmatrix} \right).
\]

Finally, we have \( L_1 \cong t(M) \) and:

\[
\vartheta_1 : t(M) \longrightarrow L_1, \quad \vartheta_1(\lambda) \longrightarrow \rho_1'(\lambda).
\]

If \( F \) is a left \( D \)-module, then applying the functor \( \text{hom}_{D}(\cdot, F) \) to the isomorphism \( E_1 \cong L_1 \), and using Malgrange’s theorem (see Theorem 1.1 of [12]), we obtain:

\[
\text{ker} F(Q_1) \cong \text{ker} F(P_1).
\]

More precisely, using (24), we get the following corollary.

**Corollary 2.2:** If \( F \) is a left \( D \)-module, then we have

\[
\text{ker} F(Q_1) \cong \text{ker} F(P_1),
\]

and the following equivalence

\[
\left\{ \begin{array}{c}
R_{11}' \theta = 0, \\
R_{21}' \theta = 0,
\end{array} \right. \iff \left\{ \begin{array}{c}
F_{12}' \tau_1 - \tau_2 = 0, \\
R_{11}' \tau_1 = 0, \\
R_{21}' \tau_1 = 0, \\
F_{13}' \tau_2 - \tau_3 = 0, \\
R_{12}' \tau_2 = 0, \\
R_{22}' \tau_2 = 0, \\
R_{13}' \tau_3 = 0, \\
R_{23}' \tau_3 = 0,
\end{array} \right.
\]
\begin{align*}
D^1 \times (p_{12} + p_{22}) & \xrightarrow{\alpha} \quad \frac{(R_{11}^{T} R_{21}^{T})^T}{(R_{11}^{T} R_{12}^{T})^T} \quad \frac{D^1 \times p_{12}}{\phi_1} \quad \frac{\rho_1}{L_2} \quad \longrightarrow 0 \\
D^1 \times (p_{11} + p_{21}) & \xrightarrow{\psi_1} \quad \frac{(R_{11}^{T} R_{21}^{T})^T}{(R_{11}^{T} R_{12}^{T})^T} \quad \frac{D^1 \times p_{11}}{\phi_2} \quad \frac{\rho_2}{L_1} \quad \longrightarrow 0.
\end{align*}

(20)

\begin{align*}
& 0 \\
& \downarrow
\begin{array}{c}
D^1 \times p_{11} \quad R_{11}' \\
D^1 \times p_{21} \quad R_{21}'
\end{array}

\begin{array}{c}
D^1 \times (p_{12}' + p_{11} + p_{21}') \\
\phi_1 \mapsto \phi_2^{-1}
\end{array}

\begin{array}{c}
E_2 \\
\pi_{21} \circ \phi_2^{-1}
\end{array}

\begin{array}{c}
L_1 \\
\theta_1 \mapsto \text{coker } \pi_{21} 
\end{array}

(23)

\begin{align*}
P \in D^s \times (p_{01} + p_{12} + p_{13} + p_{23}) & \text{ be defined by }
\begin{pmatrix}
R_{11}' & -I_{p_{11}'} & 0 & 0 \\
0 & F_{12}' & -I_{p_{12}'} & 0 \\
0 & 0 & R_{13}' & 0 \\
0 & 0 & 0 & R_{23}'
\end{pmatrix},
\end{align*}

and let us consider the following left \( D \)-modules:
\begin{align*}
M & = \frac{D^1 \times p_{01}}{(D^1 \times p_{11} R_{11})}, \\
E & = \frac{D^1 \times (p_{01} + p_{12} + p_{13})}{(D^1 \times s P)}.
\end{align*}

If \( \varphi : D^1 \times (p_{01} + p_{12} + p_{13}) \longrightarrow E \) is the canonical projection, then we have \( E \cong M \),

where the left \( D \)-isomorphism is defined by
\begin{align*}
\phi : M & \longrightarrow E \\
\pi(\lambda) & \longmapsto \varphi(\lambda (I_{p_{01}} 0 0)),
\end{align*}
\begin{align*}
\phi : E & \longrightarrow M \\
\varphi(\epsilon) & \longmapsto \pi \left( \epsilon \left( \begin{array}{cc}
I_{p_{01}} & R_{11}' \\
F_{12}' & R_{13}'
\end{array} \right) \right),
\end{align*}
(28)

Using (10), we note that the row of \( P \) containing the matrix \( R_{11}' \) can be removed. We get the following corollary of Theorem 2.1.

Corollary 2.3: With the hypotheses of Proposition 2.1 and the previous notations, if
\begin{align*}
t = s - p_{11} & = p_{11}' + p_{12}' + p_{13}' + p_{22}' + p_{13} + p_{23}', \\
st & = p_{11}' + p_{12}' + p_{21}' + p_{13}' + p_{12}' + p_{22}' + p_{13} + p_{23}',
\end{align*}

Theorem 2.1: With the hypotheses of Proposition 2.1 and the previous notations, let

\begin{align*}
s = r + p_{11} & = p_{11}' + p_{12}' + p_{11}' + p_{13}' + p_{12}' + p_{13} + p_{22}' + p_{23}',
\end{align*}

Then, the short exact sequence
\begin{align*}
0 & \longrightarrow t(M) \xrightarrow{\iota} M \xrightarrow{\rho} M/ t(M) \longrightarrow 0
\end{align*}
and $Q \in D^{1 \times (p_0, p_1, p_2, p_3)}$ is the matrix defined by

$$Q = \begin{pmatrix}
R_{11}' & -I_{p_1}' & 0 & 0 \\
0 & F_{12}' & -I_{p_2}' & 0 \\
0 & R_{21}' & 0 & 0 \\
0 & 0 & F_{13}' & -I_{p_3}' \\
0 & 0 & R_{12}' & 0 \\
0 & 0 & 0 & R_{13}' \\
0 & 0 & 0 & R_{23}'
\end{pmatrix},$$

then $M = D^{1 \times p_0} / (D^{1 \times p_1})$ is isomorphic to $M \cong E = D^{1 \times (p_0, p_1, p_2, p_3)} / (D^{1 \times 1})$, where the isomorphism is defined by (28).

Corollary 2.3 were implemented by the author in the OreModules package [7] (Maple) called PurityFiltration [13]. See also the recent homalg package [1] (GAP4) called AbelianSystems [3] developed by Barakat and the author. The AbelianSystems package is more efficient than the first algorithm implemented in homalg based on the computation of spectral sequences [2], [4], [5].

If $\mathcal{F}$ is a left $D$-module, then applying the functor $\text{hom}_D(\cdot, \mathcal{F})$ to $M \cong E$, and using Malgrange’s theorem, we obtain $\ker(\mathcal{F}(R_{11})) \cong \ker(\mathcal{F}(P)) = \ker(\mathcal{F}(Q))$. More precisely, using (28), we get the following corollary.

**Corollary 2.4:** If $\mathcal{F}$ is a left $D$-module, then we have

$$\ker(\mathcal{F}(R_{11})) \cong \ker(\mathcal{F}(P)) = \ker(\mathcal{F}(Q)),$$

and the following system equivalence holds

$$R_{11} \eta = 0 \iff \begin{cases} R_{11}' \zeta - \tau_1 = 0, \\
F_{12}' \tau_1 - \tau_2 = 0, \\
R_{21}' \tau_2 = 0, \\
F_{13}' \tau_2 - \tau_3 = 0, \\
R_{12}' \tau_3 = 0, \\
R_{22}' \tau_3 = 0, \\
R_{13}' \tau_3 = 0, \\
R_{23}' \tau_3 = 0,
\end{cases}$$

under the following invertible transformations:

$$
\begin{pmatrix}
\zeta \\
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix} \mapsto
\begin{pmatrix}
\zeta \\
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix} 
to \eta = \zeta,
$$

$$
\begin{pmatrix}
I_{p_0} \\
R_{11}' \\
F_{12}' R_{11}'' \\
F_{13}' R_{11}''
\end{pmatrix} \eta 
\mapsto \begin{pmatrix}
I_{p_0} \\
R_{11}' \\
F_{12}' R_{11}'' \\
F_{13}' R_{11}''
\end{pmatrix} \eta.
$$

**Definition 2.1 ([5], [10]):** A ring $D$ is a Cohen-Macaulay ring if $D$ is a noetherian ring equipped with a dimension function $\dim_D(\cdot)$ [10] such that:

$$\text{codim}_D(M) \triangleq \dim_D(D) - \dim_D(M) = j_D(M) \triangleq \min\{i \geq 0 \mid \text{ext}_D^i(M, D) \neq 0\}.
$$

**Example 2.1:** If $k$ is a field (resp., a field of characteristic 0), then the ring $D = k[x_1, \ldots, x_n]$ (resp., $D = A_n(k)$, $B_n(k)$, $D_n(k)$, $D_n(k)$) is a Cohen-Macaulay ring with $\dim_D(D) = n$ (resp., $\dim_D(D) = 2n$, $\dim_D(D) = 2n$, $\dim_D(D) = 2n$) [4], [5].

**Proposition 2.3 ([5])**: If $D$ is an Auslander regular ring, then $j_D(\text{ext}_D^i(M, D)) \geq i$ for all $i \in \mathbb{N}$ and for all left $D$-module $M$.

If $D$ is a Cohen-Macaulay ring, using Theorem 3.1 of [12] and Proposition 2.3, then the left $D$-modules $\text{ext}_D^i(N_{33}, D)$, $\coker \gamma_{32}$, $\coker \gamma_{21}$ and $M/t(M)$ are either 0 or satisfy:

$$\begin{cases}
\dim_D(\text{ext}_D^i(N_{33}, D)) \leq \dim_D(D) - 3, \\
\dim_D(\coker \gamma_{32}) = \dim_D(D) - 2, \\
\dim_D(\coker \gamma_{21}) = \dim_D(D) - 1, \\
\dim_D(M/t(M)) = \dim_D(D).
\end{cases}$$

**Remark 2.2:** If $S_0 = R_{11}'$ and

$$S_1 = \begin{pmatrix}
F_{12}' \\
R_{11}'' \\
R_{21}'
\end{pmatrix}, \quad S_1' = \begin{pmatrix}
F_{12}' \\
R_{11}' \\
R_{21}'
\end{pmatrix},$$

$$S_2 = \begin{pmatrix}
F_{13}' \\
R_{12}'' \\
R_{22}'
\end{pmatrix}, \quad S_3 = \begin{pmatrix}
F_{13}' \\
R_{13}' \\
R_{23}'
\end{pmatrix},$$

then using (31), we get:
1) The linear system \( \ker \phi(S_3) \cong \hom D(L_3, \mathcal{F}) \cong \hom D(\text{ext} D (N_{33}, D), \mathcal{F}) \) is either 0 or has dimension \( \leq \dim D(D) - 3 \).

2) The linear system \( \ker \phi(S_2) \cong \hom D(\text{coker } \phi(S_2), \mathcal{F}) \cong \hom D(\text{coker } \phi(S_2), \mathcal{F}) \) has dimension \( \dim D(D) - 2 \) when it is non-trivial.

3) The linear system \( \ker \phi(S_1) = \ker \phi(S_1') \cong \hom D(\text{coker } \phi(S_1), \mathcal{F}) \cong \hom D(\text{coker } \phi(S_1), \mathcal{F}) \) is either 0 or has dimension \( \dim D(D) - 1 \).

4) The linear system \( \ker \phi(S_0) \cong \hom D(M/t(M), \mathcal{F}) \) has dimension \( \dim D(D) \) when it is non-trivial.

If \( R_3 \) has full row rank, i.e., \( \ker D(R_3) = 0 \), then \( N_{33} \cong \text{ext} D(M, D) \), and thus \( \text{ext} D(N_{33}, D) \cong \text{ext} D(M, D) \), and Theorem 4.1 of [12] yields
\[
\dim D(\text{ext} D(M, D)) = \dim D(D) - 3, \]
which shows that \( \ker \phi(S_3) \) is either 0 or has dimension \( \dim D(D) - 3 \).

If \( \mathcal{F} \) is an injective \( D \)-module, then the linear system \( \ker \phi(R) = \ker \phi(R_{11}) \) can be obtained by integrating the linear system \( \ker \phi(Q) \), i.e., by integrating in casdce the linear system \( \ker \phi(S_3) \) of dimension less or equal to \( \dim D(D) - 3 \), and then the inhomogeneous linear systems of dimension respectively \( \dim D(D) - 2 \), \( \dim D(D) - 1 \) and \( \dim D(D) \). Finally, \( \ker \phi(R_{11}) = R_{01} \mathcal{F}^{v-1} \) (see [12]).

Even if the size of the matrix \( Q \) is larger than the one of \( R = R_{11} \), the matrix \( Q \) is generally more suitable for a fine study of the module properties of \( M \cong E \) than \( R \), i.e., for the study of the structural properties of the linear system \( \ker \phi(R) \cong \ker \phi(Q) \). This new form is particularly interesting for the computation of Monge parametrisations [15], [16], [12] of the linear system \( \ker \phi(R) \). Many under/overdetermined linear PD systems \( \ker \phi(R) \), which cannot directly be integrated by computer algebra systems such as Maple, can be integrated by means of their equivalent forms \( \ker \phi(Q) \) using the PURITYFILTRATION [13], AbelianSystems [3] and homalg [2] packages.

**Example 2.2:** Let us consider the \( D = \mathbb{Q}[\partial_1, \partial_2, \partial_3] \)-module \( M = D^1 \times R^4 / (D^1 \times R^6) \) finitely presented by:
\[
R = \begin{pmatrix}
0 & -2 \partial_1 & \partial_3 - 2 \partial_2 - \partial_1 & -1 \\
0 & -2 \partial_1 & 2 \partial_2 - 3 \partial_3 & 1 \\
0 & -6 \partial_1 & -2 \partial_2 - 5 \partial_1 & -1 \\
0 & \partial_3 - \partial_1 & \partial_2 - \partial_1 & 0 \\
\partial_3 & -\partial_1 & -\partial_2 - \partial_1 & 0 \\
\partial_3 - \partial_1 & -\partial_2 & 0 & 0
\end{pmatrix}
\]

Using the PURITYFILTRATION package [13], we obtain that \( M \cong E = D^1 \times 1 / (D^1 \times 15) Q \), where \( Q \) is defined by (32). Let \( \mathcal{F} = C_\infty (R^3) \) be the injective \( D \)-module (see [11] or Example 2.3 of [12]) and let us explicitly compute \( \ker \phi(Q) \).

We first integrate the last diagonal block of \( Q \), i.e., the 0-dimensional linear PD system \( \ker \phi(R_{13}) \):
\[
\begin{align*}
-\partial_2 \tau_3 &= 0, \\
-\partial_3 \tau_3 &= 0, \\
\partial_1 \tau_3 &= 0,
\end{align*}
\]

Then, we integrate the inhomogeneous linear PD system in \( \tau_2 = (\tau_{21} \quad \tau_{22} \quad \tau_{23})^T \) and \( \tau_3 \), formed by the third triangular block of \( Q \), namely:
\[
\begin{align*}
\tau_{23} - \tau_3 &= 0, \\
\tau_{21} &= 0, \\
-\tau_{21} + (4 \partial_1 - \partial_3) \tau_{22} &= 0, \\
\tau_{21} + (4 \partial_1 - \partial_3) \tau_{22} + \partial_3 \tau_{23} &= 0, \\
(\partial_1 - \partial_2) \tau_{22} &= 0, \\
\tau_{23} &= \tau_3 = c_1, \\
\tau_{21} &= 0, \\
(4 \partial_1 - \partial_3) \tau_{22} &= 0, \\
(\partial_1 - \partial_2) \tau_{22} &= 0.
\end{align*}
\]

We obtain \( \tau_{21} = 0, \tau_{22} = f_1(x_3 + \frac{1}{4} (x_1 + x_2)), \) where \( f_1 \) is an arbitrary smooth function, and \( \tau_{23} = c_1, \) where \( c_1 \) is an arbitrary constant. Then, we have to integrate the inhomogeneous linear PD system in \( \tau_1 = (\tau_{11} \quad \tau_{12} \quad \tau_{13})^T \) and \( \tau_2 \) formed by the second triangular block of \( Q \):
\[
\begin{align*}
-2 \partial_1 \tau_{12} + \tau_{13} - \tau_{21} &= 0, \\
-\tau_{12} - \tau_{22} &= 0, \\
\tau_{11} - \tau_{12} - \tau_{23} &= 0, \\
\tau_{12} &= \tau_{22} + \tau_{23} = -f_1(x_3 + \frac{1}{4} (x_1 + x_2)), \\
\tau_{13} &= -2 \partial_1 \tau_{22} + \tau_{21} = -\frac{1}{2} f_1(x_3 + \frac{1}{4} (x_1 + x_2)).
\end{align*}
\]

The entries of \( \tau_1 = (\tau_{11} \quad \tau_{12} \quad \tau_{13})^T \) are 1-dimensional and not 2-dimensional. This result comes from the fact that the matrix \( S'_1 \) (or \( S_1 \)) defined in Remark 2.2 admits a left inverse over \( D \), i.e., \( \text{coker } \phi(S_{21}) = 0 \), which yields \( \ker \phi(S'_1) \cong \hom D(\text{coker } \phi(S_{21}), \mathcal{F}) \cong \hom D(\text{coker } \phi(S_{21}), \mathcal{F}) = 0 \). Finally, we have to integrate the inhomogeneous linear PD system in \( \zeta = (\zeta_1 \ldots \zeta_4)^T \) and \( \tau_1 \), formed by the first triangular block of \( Q \), namely:
\[
\begin{align*}
\zeta_1 - \zeta_3 - \tau_{11} &= 0, \\
\zeta_2 + \zeta_3 - \tau_{12} &= 0, \\
(\partial_1 - 2 \partial_2 + \partial_3) \zeta_3 - \zeta_4 - \tau_{13} &= 0, \\
\zeta_1 - \zeta_2 &= -f_1(x_3 + \frac{1}{4} (x_1 + x_2)) + c_1, \\
\zeta_2 + \zeta_3 &= -f_1(x_3 + \frac{1}{4} (x_1 + x_2)), \\
(\partial_1 - 2 \partial_2 + \partial_3) \zeta_3 - \zeta_4 - \tau_{13} &= -\frac{1}{2} f_1(x_3 + \frac{1}{4} (x_1 + x_2)).
\end{align*}
\]

Using the fact that \( \mathcal{F} \) is an injective \( D \)-module and \( M/t(M) = D^1 \times R^4 / (D^1 \times 15 \mathcal{F}) \) is a torsion-free \( D \)-module, where \( R_{11} \) is the matrix formed by the first 3 rows and the first 4 columns of \( Q \), \( \ker \phi(R_{11}^t) \) is then parametrized by \( R_{11} = (1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2) \), i.e., \( \ker \phi(R_{11}^t) = R_{11} \mathcal{F} \) (see [6] or 1 of Corollary 2.1 of [12]). Moreover, the matrix \( R_{11} \) admits the following right inverse
\[
X = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
which implies that $M/t(M)$ is a projective $D$-module [6], and thus $M/t(M)$ is a free $D$-module of rank 1 by the Quillen-Suslin theorem [6], [9], [17]. Hence, Remark 14 of [16] (see also Remark 8 of [15]) proves that the general $F$-solution of (33) is defined by $\zeta = R_Q \xi + X \tau$, i.e.,

$$
\begin{align*}
\zeta_1 &= \xi - f_1(x_3 + \frac{1}{2} (x_1 + x_2)) + c_1, \\
\zeta_2 &= -\xi - f_1(x_3 + \frac{1}{2} (x_1 + x_2)), \\
\zeta_3 &= \xi, \\
\zeta_4 &= (\partial_1 - 2 \partial_2 + \partial_3) \xi + \frac{1}{2} f_1(x_3 + \frac{1}{2} (x_1 + x_2)),
\end{align*}
$$

for all $\xi \in C^{\infty}(\mathbb{R}^3)$, all $f_1 \in C^{\infty}(\mathbb{R})$ and all $c_1 \in \mathbb{R}$. Using the $D$-isomorphism $\gamma$ defined by (30), we finally get

$$
\begin{align*}
&-2 \partial_1 \eta_2 + \partial_3 \eta_3 - 2 \partial_2 \eta_3 - \partial_1 \eta_3 - \eta_4 = 0, \\
&\partial_3 \eta_2 - 2 \partial_1 \eta_2 + 2 \partial_2 \eta_3 - 3 \partial_1 \eta_3 + \eta_4 = 0, \\
&\partial_3 \eta_1 - 6 \partial_1 \eta_2 - 2 \partial_2 \eta_3 - 5 \partial_1 \eta_3 - \eta_4 = 0, \\
&\partial_2 \eta_2 - \partial_1 \eta_2 + 2 \partial_2 \eta_3 - \partial_1 \eta_3 = 0, \\
&\partial_2 \eta_1 - \partial_1 \eta_2 - 2 \partial_2 \eta_3 - \partial_1 \eta_3 = 0, \\
&\partial_1 \eta_1 - \partial_1 \eta_2 - 2 \partial_1 \eta_3 = 0,
\end{align*}
$$

$$
\begin{align*}
\eta_1 &= \xi - f_1(x_3 + \frac{1}{2} (x_1 + x_2)) + c_1, \\
\eta_2 &= -\xi - f_1(x_3 + \frac{1}{2} (x_1 + x_2)), \\
\eta_3 &= \xi, \\
\eta_4 &= (\partial_1 - 2 \partial_2 + \partial_3) \xi + \frac{1}{2} f_1(x_3 + \frac{1}{2} (x_1 + x_2)),
\end{align*}
$$

where $\xi$ (resp., $f_1, c_1$) is an arbitrary function of $C^{\infty}(\mathbb{R}^3)$ (resp., $C^{\infty}(\mathbb{R})$, constant).

Finally, using the regular patterns of the matrix $P$ and of (28), we can easily generalize Theorem 2.1, Corollary 2.4 and Remark 2.2 when $\ker_D(R_3) \neq 0$, i.e., for a finitely presented left $D$-module $M = D^{1 \times p_0}/(D^{1 \times p_1} R_1)$ defined by a longer finite free resolution of the form:

$$
0 \leftarrow M \xrightarrow{\pi} D^{1 \times p_0} \xrightarrow{R_1} D^{1 \times p_1} \xrightarrow{R_2} \cdots \xrightarrow{R_m} D^{1 \times p_m},
$$

If $\ker_D(R_m) = 0$, then the corresponding generalization defines the purity filtration of the left $D$-module $M$ [12].

REFERENCES