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REVISITING THE IISS SMALL-GAIN THEOREM THROUGH TRANSIENT PLUS ISS SMALL-GAIN REGULATION

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ABSTRACT

Recently, the small-gain theorem for input-to-state stable (ISS) systems has been extended to the class of integral input-to-state stable (iISS) systems. Feedback connections of two iISS systems are robustly stable with respect to disturbance if an extended small-gain condition is satisfied. It has been proved that at least one of the two iISS subsystems needs to be ISS for guaranteeing globally asymptotic stability and iISS of the overall system. Making use of this necessary condition for the stability, this paper gives a new interpretation to the iISS small gain theorem as transient plus ISS small-gain regulation. The observation provides useful information for designing and analyzing nonlinear control systems based on the iISS small-gain theorem.

Key Words: Integral input-to-state stability, nonlinear interconnected systems, small gain theorem.

I. INTRODUCTION

For analysis and design of nonlinear systems the ISS small-gain theorem has been widely used [12,17]. The theorem covers the class of input-to-state stable (ISS) systems and answers the question of whether their feedback interconnection is again ISS. It was first proved with a trajectory-based approach in [12]. A version relying on the Lyapunov functions associated with each of the subsystems was subsequently presented in [11]. While the construction of a Lyapunov function for the overall interconnection is useful from the analysis and design viewpoints, the trajectory-based proof is simpler and illustrates more intuitively the idea of contraction. Recently, the small-gain theorem has been extended to the interconnection of integral input-to-state stable (iISS) systems in [6,10]. The iISS is a more general robustness property than ISS [14,15], and the theorem in [6,10] includes the ISS small-gain theorem as a special case. In these references, a Lyapunov function is explicitly constructed for the overall interconnection. Another approach, developed in [1], makes use of monotonicity and nullclines in verifying that the equilibrium of the interconnection of iISS systems is globally asymptotically stable (GAS). Although the approach proposed there does not apply to systems with exogenous inputs, it offers a useful interpretation of GAS for feedback connections of iISS and ISS subsystems.

This paper revisits the iISS small-gain theorem and gives an insight into its mechanism. It also gives an interpretation which connects the iISS small-gain theorem with the contractive behavior of trajectories explained by the standard ISS small-gain theorem. In order to understand how the trajectories of interconnected systems evolve, this paper assumes that iISS dissipation inequalities are given for both individual subsystems. As illustrated by the result in [5] on cascaded iISS systems, the use of dissipation inequalities of subsystems is more successful than using trajectory bounds when dealing with interconnected iISS systems. This paper follows this idea to tackle feedback interconnected systems. The proof this paper develops splits the system trajectory into a transient response and a subsequent response governed by the ISS small-gain condition. This paper illustrates how this strategy enables us to deal with iISS systems which are not ISS.

Notations. The Euclidean norm of a real vector on $\mathbb{R}^n$ is denoted by the symbol $|\cdot|$. A continuous function $\omega : \mathbb{R}_+ := [0,\infty) \to \mathbb{R}_+$ is said to be positive definite and denoted by $\omega \in \mathcal{P}$ if it satisfies $\omega(0) = 0$ and $\omega(s) > 0$ holds for all $s > 0$. A function is of class $\mathcal{K}$ if it belongs to $\mathcal{P}$ and is strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$, and $\beta(s,\cdot)$ is non-increasing and goes to zero as $t \to \infty$ for each $s \geq 0$. The identity map on $\mathbb{R}$ is denoted by **Id**. If ω is a class $\mathcal{K}_\infty$ function, its inverse ω^{-1} is of class $\mathcal{K}_\infty$. For $\omega \in \mathcal{K}\backslash\mathcal{K}_\infty$, its inverse ω^{-1} is defined on the finite interval $[0,\lim_{\tau\to\infty}\omega(\tau))$ since the continuous function ω is strictly increasing and $\omega(0) = 0$. Following the convention employed in [6, 10], in this paper, $s < \lim_{\tau\to\infty}\omega(\tau)$ or

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$\infty = \lim_{\tau\to\infty}\omega(\tau)$ is implied whenever $\omega^{-1}(s)$ is used. For a function $\gamma\in\mathcal{P}$, we write $\gamma\in\mathcal{O}(>L)$ with a non-negative number L if there exists a positive number $K > L$ such that $\limsup_{s\to 0}\gamma(s)/s^K < \infty$ holds. We write $\gamma\in\mathcal{O}(L)$ when $K = L$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. For $f,g : \mathbb{R}_+ \to \mathbb{R}_+$, we use the simple notation $\lim f(s) = \lim g(s)$ to describe $\{\lim f(s) = \infty \wedge \lim g(s) = \infty\} \vee \{\infty > \lim f(s) = \lim g(s)\}$. Note that the ∞ case is included. In a similar manner, $\lim f(s) \geq \lim g(s)$ denotes $\{\lim f(s) = \infty \vee \infty > \lim f(s) \geq \lim g(s)\}$. A system $\dot{x} = f(x)$ admitting a unique maximal solution $x(t) \in \mathbb{R}^n$ for any initial condition $x(0) \in \mathbb{R}^n$ is said to be GAS if its origin is globally asymptotically stable. We let $\mathcal{U}$ denote the set of all measurable locally essentially bounded signals $u:\mathbb{R}_+\to\mathbb{R}^m$. A system $\dot{x} = f(x,u)$ admitting a unique solution $x(t)$ on $\mathbb{R}^n$ for any initial condition $x(0) \in \mathbb{R}^n$ and any $u \in \mathcal{U}$ is said to have the Bounded Energy Frequently Bounded State (BEFBS, [2]) property with respect to input u and state x if there exists $\sigma \in \mathcal{K}_\infty$ such that, if $\int_0^\infty \sigma(|u(\tau)|)d\tau < \infty$ then $\liminf_{t\to\infty}|x(t)| < \infty$ for all initial conditions $x(0)$. We will make a slight abuse of sup, limsup and inf, liminf to mean the essential supremum and infimum, respectively, where appropriate. A system $\dot{x} = f(x,u)$ is said to be iISS with respect to u if there exist $\chi\in\mathcal{K}_\infty$, $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}$ such that, for any $x(0) \in \mathbb{R}^n$ and any $u \in \mathcal{U}$, a unique solution $x(t) \in \mathbb{R}^n$ exists for all $t \geq 0$ and furthermore it satisfies $\chi(|x(t)|) \leq \beta(|x(0)|,t) + \int_0^t \gamma(|u(\tau)|)d\tau$. A system $\dot{x} = f(x,u)$ is said to be ISS with respect to u if there exist $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}$ such that, for any $x(0) \in \mathbb{R}^n$ and any $u \in \mathcal{U}$, a unique solution $x(t) \in \mathbb{R}^n$ exists for all $t \geq 0$ and furthermore it satisfies $|x(t) \leq \beta(|x(0)|,t) + \gamma(\sup_{\tau\in[0,t]}|\mu(\tau)|)$. These are standard definitions borrowed from [14,15,3].

A preliminary version of the material in this paper was presented at the 49th IEEE Conference on Decision and Control [9]. Some errors have been corrected, and the results are refined further in this paper.

II. A REVIEW OF IISS SMALL-GAIN THEOREM

Consider the following interconnected system:

$$\Sigma : \begin{cases} \Sigma_1 : \dot{x}_1 = f_1(x_1, x_2, r_1) \\ \Sigma_2 : \dot{x}_2 = f_2(x_1, x_2, r_2) \end{cases} \tag{1}$$

where $x_i(t)\in\mathbb{R}^{n_i}$, $r_i(t)\in\mathbb{R}^{m_i}$, $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$ and $r = [r_1^T, r_2^T]^T \in \mathbb{R}^m$. In addition to the existence of a unique maximal solution $x(t)$ for any initial condition $x(0) \in \mathbb{R}^n$ and any measurable, locally essentially bounded external input r, we assume that the two subsystems satisfy the following dissipation inequalities:

Assumption 1. For each $i \in \{1,2\}$, there exist a continuously differentiable, positive definite and radially unbounded function V_i: $x_i \in \mathbb{R}^{n_i} \to \mathbb{R}_+$ and class $\mathcal{K}$ functions α_i, σ_i, σ_{ri} such that

$$\dot{V}_1 \leq -\alpha_1(V_1(x_1)) + \sigma_1(V_2(x_2)) + \sigma_{r1}(|r_1|) \tag{2}$$

$$\dot{V}_2 \leq -\alpha_2(V_2(x_2)) + \sigma_2(V_1(x_1)) + \sigma_{r2}(|r_2|) \tag{3}$$

hold for all $r \in \mathcal{U}$ along the trajectories $x(t)$ of (1).

This assumption imposes that each subsystem Σ_i is iISS with respect to input (x_{3-i}, r_i) and state x_i (see, for instance [3]). We stress that we have assumed $\alpha_i \in \mathcal{K}$ instead of $\alpha_i \in \mathcal{P}$ without any loss of generality due to the necessity result in [7] for iISS feedback connections with $\sigma_i \in \mathcal{K}$. In the case of cascade, assuming $\alpha_i \in \mathcal{K}$ is not necessary [4,5].The following is a result in [10], which is referred to as the iISS small-gain condition in this paper.

Theorem 1. Suppose that Assumption 1 holds and that there exist ω_1, $\omega_2 \in \mathcal{K}_\infty$ satisfying

$$\alpha_1^{-1}\circ(\mathbf{Id}+\omega_1)\circ\sigma_1\circ\alpha_2^{-1}\circ(\mathbf{Id}+\omega_2)\circ\sigma_2(s) \leq s, \quad \forall s\in\mathbb{R}_+. \tag{4}$$

Then, the following statements hold true:

(i) For $r(t) \equiv 0$, the system (1) is GAS.
(ii) If it holds that

$$\{\lim_{s\to\infty}\alpha_i(s) = \infty \vee \lim_{s\to\infty}\sigma_{3-i}(s) < \infty\}, \quad i = 1, 2, \tag{5}$$

then the system (1) is iISS with respect to input r and state x.

It is strongly stressed that the small-gain condition (4) with the existence of ω_1, $\omega_2 \in \mathcal{K}_\infty$ implicitly requires that

$$\lim_{s\to\infty}\alpha_2(s) = \infty \vee \lim_{s\to\infty}\alpha_2(s) > \lim_{s\to\infty}\sigma_2(s). \tag{6}$$

It is also important that the property (6) implies that Σ_2 is ISS with respect to its feedback input x_1 (see for instance [16,3]). On the other hand, Σ_1 does not have to be ISS with respect to its feedback input x_2. The small-gain condition for iISS subsystems indicates that the interconnection is stable if the stability property of one subsystem, Σ_2, is "strong" enough to compensate the "weak stability" of the other subsystem, Σ_1. Due to this asymmetry, we need to select or interchange the indices "1" and "2" so that (4) holds when iISS subsystems are involved. The condition (4) reduces to the one for the ISS small-gain theorem [12,11] when α_1, $\alpha_2 \in \mathcal{K}_\infty$. We have (5) since the nonlinear gain $\alpha_i^{-1}\circ(\mathbf{Id}+\omega_i)\circ\sigma_i$ of the subsystem Σ_i is computed independently of the influence of r_i from the dissipative inequality (2) or (3). The necessity of the condition (6) for stability of the interconnected system is investigated in [10], which is summarized as follows:

Theorem 2. Suppose that $\alpha_i \in \mathcal{O}(1)$ and $\sigma_i \in \mathcal{O}(>0)$ which are continuously differentiable on $(0, \infty)$ are given for $i = 1,2$. Then, the following statements hold true:

(i) The system (1) with $r(t) \equiv 0$ is GAS for all subsystems satisfying Assumption 1 only if

$$\lim_{s\to\infty}\alpha_j(s) \geq \lim_{s\to\infty}\sigma_j(s) \tag{7}$$

holds for at least one of $j \in \{1,2\}$.

(ii) The system (1) is ISS with respect to input r and state x for all subsystems satisfying Assumption 1 only if

$$\lim_{s\to\infty}\alpha_j(s) = \infty \vee \lim_{s\to\infty}\alpha_j(s) > \lim_{s\to\infty}\sigma_j(s) \tag{8}$$

holds for at least one of $j \in \{1,2\}$.

Renumbering allow us to take $j = 2$ for (7) and (8) without any loss of generality. This convention is used in the rest of this paper. The above theorem does not exactly state that (6) is necessary for the iISS of the interconnection. The difference between (6) and (7) is the equality. The main body of this paper does not address the equality case

$$\infty > \lim_{s\to\infty}\alpha_2(s) = \lim_{s\to\infty}\sigma_2(s) \tag{9}$$

since it formally prevents us from using the ISS small-gain argument [12,11]. Notice that the equality case (9) is incompatible with the small gain condition (4) defined with $\omega_1, \omega_2 \in \mathcal{K}_\infty$. Taking into account the necessity of (6) for the ISS case, this paper assumes (6) and makes use of it for proving Theorem 1 in order to interpret the "iISS" small-gain theorem as the combination of "a transient response" and "the ISS small-gain dynamics". In order to give the new interpretation, this paper makes another assumption on the influence of the exogenous signals r_1 and r_2, *i.e.*, Assumption 2, which is the fundamental limitation of the idea of resorting to the "ISS" small-gain argument for "non-ISS" systems.

Remark 1. The necessary conditions in Theorem 2 were proved for $\alpha_i \in \mathcal{O}(>1)$ in [10]. It can be verified that $\alpha_i \in \mathcal{O}(>1)$ can be replaced by $\alpha_i \in \mathcal{O}(1)$ for supply rates given as functions of V_1 and V_2 as in Assumption 1 (See [8]).

III. ESTABLISHING 0-GAS

This section considers the interconnected system (1) in the absence of the external signals, *i.e.*, $r(t) \equiv 0$, and demonstrates Item (i) of Theorem 1 by means of a transient response plus the ISS small-gain argument. We refer to the stability as 0-GAS of Σ. Define the following set:

$$\mathbf{U}_2 := \left\{x_2 \in \mathbb{R}^{n_2} : V_2(x_2) \leq \lim_{s\to\infty}\alpha_2^{-1}\circ\sigma_2(s)\right\}$$

Notice that $\mathbf{U}_2 := \mathbb{R}^{n_2}$ holds if $\lim_{s\to\infty}\sigma_2(s) = \infty$. The following proposition separates each trajectory of the interconnected system Σ into two phases.

Proposition 1. Suppose that Assumption 1 holds and that there exist $\omega_1, \omega_2 \in \mathcal{K}_\infty$ satisfying (4). Then for each $x_2(0) \in \mathbb{R}^{n_2}$, there exists $T \in \mathbb{R}_+$ such that

$$x_2(t) \in \mathbf{U}_2, \quad \forall t \geq T \tag{10}$$

$$\sup_{\tau\in[0,T]} |x(\tau)| < \infty. \tag{11}$$

Furthermore, the following two properties hold:

$$\begin{aligned}&V_1(x_1) \geq \alpha_1^{-1}\circ(\mathbf{Id}+\omega_1)\circ\sigma_1(V_2(x_2)) \wedge x_2 \in \mathbf{U}_2\\ &\Rightarrow \dot{V}_1 \leq -(\mathbf{Id}-(\mathbf{Id}+\omega_1)^{-1})\circ\alpha_1(V_1(x_1))\end{aligned} \tag{12}$$

$$\begin{aligned}&V_2(x_2) \geq \alpha_2^{-1}\circ(\mathbf{Id}+\omega_2)\circ\sigma_2(V_1(x_1))\\ &\Rightarrow \dot{V}_2 \leq -(\mathbf{Id}-(\mathbf{Id}+\omega_2)^{-1})\circ\alpha_2(V_2(x_2)).\end{aligned} \tag{13}$$

Proof. Assume that (4) is satisfied for some $\omega_1, \omega_2 \in \mathcal{K}_\infty$. It implies that

$$\begin{aligned}\sup_{x_2\in\mathbf{U}_2}\sigma_1(V_2(x_2)) &\leq \lim_{s\to\infty}\sigma_1\circ\alpha_2^{-1}\circ\sigma_2(s)\\ &\leq \lim_{s\to\infty}(\mathbf{Id}+\omega_1)^{-1}\circ\alpha_1(s).\end{aligned}$$

From this property and (2)–(3) it follows that the properties (12) and (13) hold. Next, suppose for the time being that

$$\lim_{s\to\infty}\alpha_1(s) < \infty \wedge \lim_{s\to\infty}\alpha_1(s) \leq \lim_{s\to\infty}\sigma_1(s). \tag{14}$$

Then, the small-gain condition (4) implies that there exists a positive constant σ_2^{max} such that

$$\lim_{s\to\infty}\sigma_2(s) \leq \sigma_2^{max} < \infty. \tag{15}$$

Since σ_2^{max} is independent of x_1, the dissipation inequality (3) of Σ_2 and the property (6) implied by (4) guarantee that the state $x_2(t)$ is bounded and eventually enters the forward invariant set $\mathbf{U}_2$, *i.e.*, (10). In fact, there exists $\delta > 0$ such that $\dot{V}_2 \leq -\delta$ holds for all $x_2 \notin \mathbf{U}_2$. Note that $T \geq 0$ fulfilling (10) is finite, and that the state $x_1(t)$ is bounded over the time interval $[0,T]$ since Σ_1 is iISS with respect to input x_2. Thus, we have (11). Finally, in the case that (14) does not hold, that is:

$$\lim_{s\to\infty}\alpha_1(s) = \infty \vee \lim_{s\to\infty}\alpha_1(s) > \lim_{s\to\infty}\sigma_1(s),$$

the subsystem Σ_1 is ISS with respect input x_2, and we can invoke the ISS small-gain argument [12,11] to obtain (10) and

(11). The time T can be made independent of $x_1(0)$ since $\mathbf{U}_2 \neq \mathbb{R}^{n_2}$ implies (15). □

The property (12) implies that Σ_1 exhibits an ISS property when the input x_2 is restricted to $\mathbf{U}_2$. Note that $\mathbf{Id} - (\mathbf{Id} + \omega_i)^{-1} \in \mathcal{K}_\infty$ since $(\mathbf{Id} - (\mathbf{Id} + \omega_i)^{-1}) \circ (s + \omega_i(s)) = \omega_i(s)$. Due to (12) and (13), the convergence of $x(t)$ to the origin $x = 0$ departing from any $x(T) \in \mathbb{R}^{n_1} \times \mathbf{U}_2$ at $t = T$ is ensured by the small-gain condition (4). For instance, we can follow the proof for the interconnection of the two ISS subsystems given in [12,11] and [17] which describes the small-gain argument with the domain restriction. Therefore, Proposition 1 yields a proof of Item (i) of Theorem 1. The behavior before $t = T$ described by (10) and (11) is transient. After the finite time $t = T$, the contractive dynamics kicks in since the iISS small-gain condition acts as the ISS small-gain condition in the domain $\mathbf{U}_2$ where the trajectories evolve.

Remark 2. In the absence of external signals, *i.e.* $r(t) \equiv 0$, a Lyapunov function establishing the GAS of the interconnected system can be constructed even when (9) holds. In fact, Theorem 1 in [10] derives such a Lyapunov function from a small-gain condition. The small-gain condition is exactly in the form of (4). However, the amplification factors ω_1, ω_2 in (4) for GAS are not necessarily of class $\mathcal{K}_\infty$ in the absence of the exogenous signal r. Although the above argument in this section does not explicitly address (9), the observation of the transient plus the ISS small-gain dynamics still holds true. Notice that the ISS small-gain theorem [12,11] applies to the case of

$$\lim_{s\to\infty} \alpha_1(s) = \lim_{s\to\infty} \sigma_1(s) \wedge \text{Eq. (9)} \tag{16}$$

directly since V_1 and V_2 become ISS Lyapunov functions of the individual subsystems. Therefore, Proposition 1 holds true even for $\omega_i \notin \mathcal{K}_\infty$ in the case of (16). If

$$\lim_{s\to\infty} \alpha_1(s) > \lim_{s\to\infty} \sigma_1(s) \wedge \text{Eq. (9)} \tag{17}$$

holds, by virtue of $\alpha_2^{-1} \circ \sigma_2 \in K_\infty$, the argument given in this section can be used by switching the indices "1" and "2". Note that the property (17) with the switching allows us to assume $\omega_1, \omega_2 \in \mathcal{K}_\infty$ for (4) there. The situation

$$\lim_{s\to\infty} \alpha_1(s) < \lim_{s\to\infty} \sigma_1(s) \wedge \text{Eq. (9)} \tag{18}$$

is excluded by theorem 5 (i) in [10]. Therefore, for the GAS case (*i.e.*, for $r(t) \equiv 0$), the interpretation of the transient plus the ISS small-gain dynamics is valid whenever

$$\lim_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} \sigma_2(s). \tag{19}$$

IV. ESTABLISHING IISS

This section proves Item (ii) of Theorem 1 under an assumption about disturbance magnitude. The property (6) implied by the small-gain condition (4) again plays a key role in implementing the idea of a transient plus the ISS small-gain argument. The proof consists of two parts. One is to verify that the system (1) is 0-GAS (that is, GAS when $r(t) \equiv 0$). The other part is to establish the Bounded Energy Frequently Bounded State (BEFBS) property of the system (1). It is shown in [2] that the combination of the above two properties is equivalent to the iISS property of the system (1). Since the 0-GAS has been proved in the previous section, this section is devoted to the BEFBS property.

First, notice that $\lim_{s\to\infty}\alpha_i(s) > \lim_{s\to\infty}\sigma_i(s)$ does not guarantee the ISS property of Σ_i with respect to input (x_{3-i}, r_i) since $\lim_{s\to\infty}\sigma_{ri}(s)$ can anyway be larger than $\lim_{s\to\infty}\alpha_i(s)$. In fact, when there exists $i \in \{1,2\}$ such that $\lim_{s\to\infty}\alpha_i(s) < \infty$ holds, the previously existing results show only the iISS of the interconnected system [6,10]. Hence, in contrast to the GAS case, the condition $\lim_{s\to\infty}\alpha_2(s) > \lim_{s\to\infty}\sigma_2(s)$ is not sufficient for resorting to the ISS small-gain argument in the presence of external inputs. Therefore, in order to make use of the small-gain argument of ISS-type, we introduce the following:

Assumption 2. The following properties hold:

$$\lim_{s\to\infty} \alpha_2(s) = \infty \vee \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \tag{20}$$

$$\begin{aligned} &\lim_{s\to\infty} \omega_1 \circ \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \\ &\quad \geq \lim_{s\to\infty} \left\{ \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2^{-1}) \circ \sigma_{r2}(s) + \sigma_{r1}(s) \right\} \end{aligned} \tag{21}$$

$$\begin{aligned} &\lim_{s\to\infty} \alpha_2(s) = \infty \vee \\ &\qquad \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} (\mathbf{Id} - (\mathbf{Id} + \omega_2)^{-1})^{-1} \circ \sigma_{r2}(s). \end{aligned} \tag{22}$$

Notice that we can pick $\omega_2 \in \mathcal{K}_\infty$ fulfilling (20) whenever there exists a pair of ω_1, $\omega_2 \in \mathcal{K}_\infty$ satisfying (4). Define:

$$\mathbf{U}_{D2} := \left\{ x_2 \in \mathbb{R}^{n_2} : V_2(x_2) \leq \lim_{s\to\infty} \alpha_2^{-1} \circ \{\sigma_2(s) + \sigma_{r2}(s)\} \right\}.$$

Let $\mathbf{U}_{D2} := \mathbb{R}^{n_2}$ if $\lim_{s\to\infty} \alpha_2(s) < \lim_{s\to\infty} \sigma_2(s) + \sigma_{r2}(s)$. The next proposition shows that the above assumption allows us to separate each trajectory of the interconnected system Σ into two phases even in the presence of the external inputs.

Proposition 2. Suppose that Assumption 1 and the property (5) hold. Assume that there exist ω_1, $\omega_2 \in \mathcal{K}_\infty$ satisfying (4) and Assumption 2. Then for each $x_2(0) \in \mathbb{R}^{n_2}$, there exists $T \in \mathbb{R}_+$ such that

$$x_2(t) \in \mathbf{U}_{D2}, \quad \forall t \geq T \tag{23}$$

$$\sup_{\tau\in[0,T]} |x(\tau)| < \infty \tag{24}$$

$$\mathbf{U}_{D2} = \mathbb{R}^{n_2} \Rightarrow T = 0 \tag{25}$$

hold for any measurable, locally essentially bounded r. Furthermore, there exist $\gamma \in \mathcal{K}_\infty$ and $w \in \mathbb{R}_+$ such that

$$\sup_{t\in[T,\infty)} |r(t)| \leq l \Rightarrow \limsup_{t\to\infty} |x(t)| \leq \gamma(l) + w \qquad \forall x_1(T) \in \mathbb{R}^{n_1}, \quad x_2(T) \in \mathbf{U}_{D2}. \tag{26}$$

Proof. First, assume that

$$\lim_{s\to\infty} \sigma_1(s) = \lim_{s\to\infty} \sigma_2(s) = \infty \tag{27}$$

is satisfied. Then the condition (4) and the implicit requirement (6) yield $\lim_{s\to\infty}\alpha_1(s) = \lim_{s\to\infty}\alpha_2(s) = \infty$ and $\mathbf{U}_2 = \mathbb{R}^{n_2}$. We obtain (23), (24) and (26) for any $T \in \mathbb{R}_+$ by using the ISS small-gain result in [12,11]. Hence, in the rest of the proof, we assume

$$\lim_{s\to\infty} \sigma_1(s) < \infty \vee \lim_{s\to\infty} \sigma_2(s) < \infty. \tag{28}$$

The definition of $x_2 \in \mathbf{U}_{D2}$ yields

$$\begin{aligned}\sup_{x_2\in\mathbf{U}_{D2}} \sigma_1(V_2(x_2)) &\leq \lim_{s\to\infty} \sigma_1 \circ \alpha_2^{-1} \circ \{\sigma_2(s) + \sigma_{r2}(s)\} \\ &\leq \lim_{s\to\infty} \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \\ &\quad + \lim_{s\to\infty} \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2^{-1}) \circ \sigma_{r2}(s)\end{aligned}$$

To derive the second inequality, the two cases separated by $\omega_2 \circ \sigma_2(s) \geq \sigma_{r2}(s)$ and $\omega_2 \circ \sigma_2(s) < \sigma_{r2}(s)$ are combined. From (2) it follows that, for all $x_2 \in \mathbf{U}_{D2}$,

$$\begin{aligned}\dot{V}_1 \leq& -\alpha_1(V_1) + \lim_{s\to\infty} \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \\ &+ \lim_{s\to\infty} \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2^{-1}) \circ \sigma_{r2}(s) + \sigma_{r1}(|r_1|).\end{aligned} \tag{29}$$

The property (21) applied to the above gives

$$\dot{V}_1 \leq -\alpha_1(V_1) + \lim_{s\to\infty} (\mathbf{Id} + \omega_1) \circ \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \tag{30}$$

Recall that (20) and (28) imply

$$\lim_{s\to\infty} (\mathbf{Id} + \omega_1) \circ \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) < \infty.$$

Applying this property and (4) to (30) and (29), we can verify that there exist a function $\gamma_1 \in \mathcal{K}_\infty$ and a constant $w_1 \geq 0$ such that

$$\sup_{t\in[T,\infty)} |r_1(t)| \leq l_1 \Rightarrow \begin{cases} \sup_{t\in[T,\infty)} |x_1(t)| < \infty \\ \limsup_{t\to\infty} |x_1(t)| \leq \gamma_1(l_1) + w_1 \end{cases} \qquad \forall x_1(T) \in \mathbb{R}^{n_1}, \quad x_2(T) \in \mathbf{U}_{D2}. \tag{31}$$

If

$$\lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s) + \sigma_{r2}(s), \tag{32}$$

is satisfied, the boundedness of $\mathbf{U}_{D2}$ and (31) yield the bounded-input bounded-state property (26) over $t \in [T, \infty)$ provided that (23) holds. In the case of

$$\lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s) + \sigma_{r2}(s) = \infty, \tag{33}$$

the set $\mathbf{U}_{D2}$ is unbounded, *i.e.*, $\mathbf{U}_{D2} = \mathbb{R}^{n_2}$. We temporarily assume that

$$\lim_{s\to\infty} \alpha_1(s) < \infty \vee \lim_{s\to\infty} \alpha_2(s) < \infty. \tag{34}$$

This property ensures $\lim_{s\to\infty}\sigma_2(s) < \infty$. Indeed, it is implied by (5) if $\lim_{s\to\infty}\alpha_1(s) < \infty$. The property (6) yields $\lim_{s\to\infty}\sigma_2(s) < \infty$ in the case of $\lim_{s\to\infty}\alpha_2(s) < \infty$. The dissipation inequality (3) of Σ_2 with $\lim_{s\to\infty}\sigma_2(s) < \infty$ guarantees the existence of a function $\gamma_2 \in \mathcal{K}_\infty$ and a constant $w_2 \geq 0$ such that

$$\sup_{t\in[T,\infty)} |r_2(t)| \leq l_2 \Rightarrow \begin{cases} \sup_{t\in[T,\infty)} |x_2(t)| < \infty \\ \limsup_{t\to\infty} |x_2(t)| \leq \gamma_2(l_2) + w_2 \end{cases} \qquad \forall x_1(T) \in \mathbb{R}^{n_1}, \quad x_2(T) \in \mathbf{U}_{D2} = \mathbb{R}^{n_2}. \tag{35}$$

Combining (31) and (35) yields (26) for an arbitrary $T \in \mathbb{R}_+$. We now retract (34) and assume that

$$\lim_{s\to\infty} \alpha_1(s) = \infty \wedge \lim_{s\to\infty} \alpha_2(s) = \infty \tag{36}$$

holds. Then the standard ISS small-gain theorem yields (23), (24) and (26) with $T = 0$ since $\mathbf{U}_{D2} = \mathbb{R}^{n_2}$.

To see that one of (32) and (33) must hold, consider (22). Then there exists $\beta \in \mathcal{K}_\infty$ such that

$$\begin{aligned}&\lim_{s\to\infty} \alpha_2(s) = \infty \vee \\ &\quad \lim_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} ((\mathbf{Id} + \beta)^{-1} - (\mathbf{Id} + \omega_2)^{-1})^{-1} \circ \sigma_{r2}(s).\end{aligned} \tag{37}$$

By virtue of (37) and

$$\lim_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} (\mathbf{Id} + \omega_2) \circ \sigma_2(s)$$

implied by (4), the property

$$\lim_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} (\mathbf{Id} + \beta) \circ (\sigma_2(s) + \sigma_{r2}(s)) \tag{38}$$

holds since

$$\lim_{s\to\infty}\sigma_2(s)+\lim_{s\to\infty}\sigma_{r2}(s)\le\lim_{s\to\infty}\big((\mathbf{Id}+\omega_2)^{-1}$$
$$+(\mathbf{Id}+\beta)^{-1}-(\mathbf{Id}+\omega_2)^{-1}\big)\circ\alpha_2(s)$$

is satisfied for $\lim_{s\to\infty}\alpha_2(s)<\infty$. Thus, we arrive at one of (32) and (33).

Next, we shall establish (23), (24) and (25). Since the case where both (33) and (36) hold has already been solved, we only need to consider the complementary case. If (33) and (34) hold, we have obtained $\mathbf{U}_{D2}=\mathbb{R}^{n_2}$ for which (31), (35) and (26) are satisfied for an arbitrary $T\in\mathbb{R}_+$. Taking $T=0$ is satisfactory for (23), (24) and (25). Consider the remaining case where (32) holds. By virtue of $\mathbf{U}_{D2}\neq\mathbb{R}^{n_2}$, the implication (25) holds true, *i.e.*, it can be skipped. The dissipation inequality (3) of Σ_2 guarantees that the state $x_2(t)$ which is bounded enters the set $\mathbf{U}_{D2}$ in a finite time $T\ge 0$ and remains there, *i.e.*, (23), and T can be picked independently of $x_1(0)$. Note that the state $x_1(t)$ is also bounded in the time interval $[0,T]$ since Σ_1 is iISS with respect to input (x_2,r_1) and state x_1. Hence, we arrive at (24). □

Proposition 2 demonstrates that, even in the presence of the external signal r, the behavior up to $t=T$ can be considered as a transient in view of (23) and (24). After $t=T$, the bounded-input bounded-state property (26) takes effect since the iISS small-gain condition acts as the ISS small-gain condition in the domain $\mathbf{U}_{D2}$ where the trajectories evolve. The bounded-input bounded-state property for $t\in[T,\infty)$ preceded by the transient for $t\in[0,T)$ implies the BEFBS property of Σ with respect to input r and state x. Note that we have $T=0$ if $\mathbf{U}_{D2}=\mathbb{R}^{n_2}$. These facts together with the 0-GAS proved in the previous section complete the proof of Item (ii) of Theorem 1.

V. ANOTHER FORMULATION FOR DISTURBANCE

In the presence of external signals, the idea of the reduction to the ISS small-gain argument in the presence of an iISS subsystem can be seen more or less in a compact manner if one uses dissipation inequalities of another type for the iISS property of the individual subsystems. To this end, in this section, we replace Assumptions 1 and 2 with the following two assumptions.

Assumption 3. For each $i\in\{1,2\}$, there exist a continuously differentiable positive definite and radially unbounded function V_i: $\mathbb{R}^{n_i}\to\mathbb{R}_+$ and class $\mathcal{K}$ functions α_i, σ_i, σ_{ri} such that

$$\dot V_1\le-\alpha_1(V_1(x_1))+\max\{\sigma_1(V_2(x_2)),\sigma_{r1}(|r_1|)\}\tag{39}$$

$$\dot V_2\le-\alpha_2(V_2(x_2))+\max\{\sigma_2(V_1(x_1)),\sigma_{r2}(|r_2|)\}\tag{40}$$

hold for all $r\in\mathcal{U}$ along the trajectories $x(t)$ of (1).

Assumption 4. The following properties hold:

$$\lim_{s\to\infty}\sigma_1\circ\alpha_2^{-1}\circ\sigma_2(s)\ge\max\lim_{s\to\infty}\{\sigma_1\circ\alpha_2^{-1}\circ\sigma_{r2}(s),\sigma_{r1}(s)\}\tag{41}$$

$$\lim_{s\to\infty}\alpha_2(s)=\infty\vee\lim_{s\to\infty}\alpha_2(s)>\lim_{s\to\infty}\sigma_{r2}(s).\tag{42}$$

Assumption 3 is quantitatively different from Assumption 1. However, they are qualitatively equivalent in view of the standard relation $a+b\le\max\{2a,2b\}\le 2a+2b$ for $a,b\in\mathbb{R}_+$. The quantitative difference in the formulation of subsystems brings in the technical difference between Assumptions 2 and 4. Indeed, when the interconnection of two iISS subsystems is defined with Assumption 3 in Theorem 1, we are able to achieve the reduction to the transient plus the ISS small-gain argument under Assumption 4 which may look more intuitive than Assumption 2. The rest of this section sketches this fact.

Since the 0-GAS property is proved in Section III, we shall prove the BEFBS property of the system (1). As in Section IV, we can assume (34). The properties (5) and (6) ensure $\lim_{s\to\infty}\sigma_2(s)<\infty$. Suppose that $\lim_{s\to\infty}\sigma_{r2}(s)<\infty$. Due to (42) and (6), the dissipation inequality (40) of Σ_2 guarantees that the state $x_2(t)$ which is bounded enters the set

$$\mathbf{U}_{M2}:=\Big\{x_2\in\mathbb{R}^{n_2}:V_2(x_2)\le\lim_{s\to\infty}\alpha_2^{-1}\circ\max\{\sigma_2(s),\sigma_{r2}(s)\}\Big\}$$

in a finite time T and stays there. When $\lim_{s\to\infty}\sigma_{r2}(s)=\infty$ holds, the same property holds with T which satisfies $T<\infty$ for each $\sup_{t\in[0,\infty)}|r_2(t)|<\infty$. The state $x_1(t)$ is also bounded for the time interval $[0,T]$ since Σ_1 is iISS with respect to input (x_2,r_1) and state x_1. By definition we have:

$$\sup_{x_2\in\mathbf{U}_{M2}}\sigma_1(V_2(x_2))\le$$
$$\max\Big\{\lim_{s\to\infty}\sigma_1\circ\alpha_2^{-1}\circ\sigma_2(s),\lim_{s\to\infty}\sigma_1\circ\alpha_2^{-1}\circ\sigma_{r2}(s)\Big\}.$$

From (39) it follows that, for all $x_2\in\mathbf{U}_{M2}$,

$$\begin{aligned}\dot V_1\le-\alpha_1(V_1(x_1))+\max\{&\lim_{s\to\infty}\sigma_1\circ\alpha_2^{-1}\circ\sigma_2(s),\\&\lim_{s\to\infty}\sigma_1\circ\alpha_2^{-1}\circ\sigma_{r2}(s),\sigma_{r1}(|r_1|)\}.\end{aligned}\tag{43}$$

Here, the property $\lim_{s\to\infty}\sigma_1\circ\alpha_2^{-1}\circ\sigma_2(s)<\infty$ holds due to $\lim_{s\to\infty}\sigma_2(s)<\infty$ and (6). Thus, the assumption (41) and the small-gain condition (4) lead us to the bounded-input bounded-state property (31) with respect to state x_1 in the interval of $t\in[T,\infty)$ for the initial conditions $x_2(T)\in\mathbf{U}_{M2}$. These facts allow us to arrive at Proposition 2 replacing Assumptions 1 and 2 with Assumptions 3 and 4. Hence, we obtain the BEFBS property of Σ, and Item (ii) of Theorem 1 is proved.

Assumption 1 and Assumption 3 are qualitatively equivalent in the sense that $\sigma_i + \sigma_{ri} \le \max\{2\sigma_i, 2\sigma_{ri}\} \le 2\sigma_i + 2\sigma_{ri}$. We can consider other variants of dissipation inequalities for iISS. Although the coefficients appearing in the transformation between two representations result in conservativeness in different forms, the essence of imposing the constraint on the external inputs for the reduction to the ISS small-gain argument remains the same.

Remark 3. As already stressed, the difficulty in establishing the iISS via the transient plus the ISS small-gain dynamics arises when the effect of r_i's is larger than the contribution of α_i's. Both Assumption 1 and Assumption 3 allow the magnitude of σ_{ri}'s to be arbitrarily large. In order to make the ISS small-gain argument work, the undesirably large effect of r_is is avoided by Assumptions 2 and 4. In short, σ_{ri}s are required to be sufficiently small in this paper. It is worth noting that the pair of ISS with respect to small inputs and forward completeness does not always imply iISS. Indeed, one can construct a forward complete non-iISS system which is ISS with respect to small inputs by modifying the technique proposed in [3, Section V]. In the presence of arbitrarily large σ_{ri}'s, removing Assumptions 2 and 4 is inherently difficult.

Remark 4. Neither the pair (21)–(22) nor the pair (41)–(42) is necessary for establishing the iISS property of the interconnection of iISS subsystems. For example, in the case where $\alpha_1 = \sigma_2$, $\alpha_2 = d\sigma_1$ with σ_{r1}, $\sigma_{r2} \in \mathcal{K}_\infty$ and some $d > 1$, the function $V = V_1 + V_2(1 + 1/d)/2$ is an iISS Lyapunov function, thus immediately proving the iISS of the interconnection. In contrast to the approach pursued in this paper, this case is covered by the iISS small-gain theorems proposed in [6,10]. Therefore, the approach based on the ISS small-gain argument plus the transient is more restrictive than the direct iISS small-gain approach.

VI. AN ILLUSTRATIVE EXAMPLE

Consider the interconnected system described by:

$$\dot{x}_1 = -\frac{x_1}{1+x_1^2} + \frac{x_1}{2(1+x_1^2)}(x_2 + r_1) \tag{44}$$

$$\dot{x}_2 = -x_2 + \frac{x_1^2}{1+x_1^2} \tag{45}$$

This pair satisfies the dissipation inequalities

$$\dot{V}_1 \le -\frac{2V_1(x_1)}{1+V_1(x_1)} + \sqrt{V_2(x_2)} + |r_1| \tag{46}$$

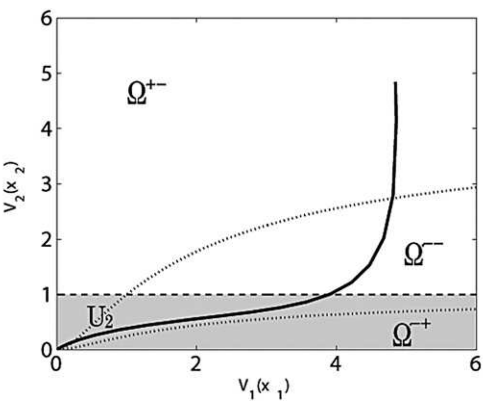

Fig. 1. The trajectory of (44)–(45) on the (V_1,V_2)-plane for $r_1(t) \equiv 0$ and $x(0) = [2.2,2.2]^T$ with relation to the sets Ω^{+-}, Ω^{--}, Ω^{-+} and $\mathbf{U}_2$.

$$\dot{V}_2 \le -V_2(x_2) + \left(\frac{V_1(x_1)}{1+V_1(x_1)}\right)^2 \tag{47}$$

for $V_1(x_1) = x_1^2$ and $V_2(x_2) = x_2^2$. Note that the upper bounds in (46) and (47), *i.e.*, the supply rates, may not be completely tight. The subsystem Σ_1 is not ISS with respect to input x_2, and it is only iISS. The trajectory of (44)–(45) for the initial condition $x(0) = [2.2,2.2]^T$ is plotted on the (V_1,V_2)-plane in Fig. 1 for $r_1(t) \equiv 0$. Fig. 1 also depicts the following sets:

$$\Omega^{+-} := \{(V_1,V_2) \in \mathbb{R}_+^2 : \alpha_1(V_1) \le \sigma_1(V_2) \wedge \alpha_2(V_2) \ge \sigma_2(V_1)\}$$
$$\Omega^{--} := \{(V_1,V_2) \in \mathbb{R}_+^2 : \alpha_1(V_1) \ge \sigma_1(V_2) \wedge \alpha_2(V_2) \ge \sigma_2(V_1)\}$$
$$\Omega^{-+} := \{(V_1,V_2) \in \mathbb{R}_+^2 : \alpha_1(V_1) \ge \sigma_1(V_2) \wedge \alpha_2(V_2) \le \sigma_2(V_1)\}.$$

The boundaries of these sets are not necessarily the nullclines of (44) and (45) owing to the lack of tightness in the dissipation inequalities (46) and (47). Two phases are observed in Fig. 1. The first phase is the transient evolving outside $\mathbf{U}_2$ for which the trajectory heads. The second phase is the trajectory converging to the origin without leaving $\mathbf{U}_2$. Once the trajectory enters the positively invariant set $\mathbf{U}_2$, the dynamic is governed by the ISS small-gain condition as discussed in Section III. It is also seen in Fig. 1 near the origin that the set Ω^{--} is too narrow to be an invariant set because of the gaps in the dissipation inequalities. Fig. 2 shows the response for the same initial condition in the presence of the disturbance $r_1(t) = 1.8/(2 + t)$. The trajectory is bounded and moves toward the set $\mathbf{U}_2$ which becomes positively invariant again. Since the iISS small-gain condition acts as the ISS small-gain condition in $\mathbf{U}_2$, we see that the trajectory converges to the origin. It conforms to the converging-input converging-state of the ISS property. The boundedness and the converging property agree with the iISS property for the entire $t \ge 0$ which is established in Section IV.

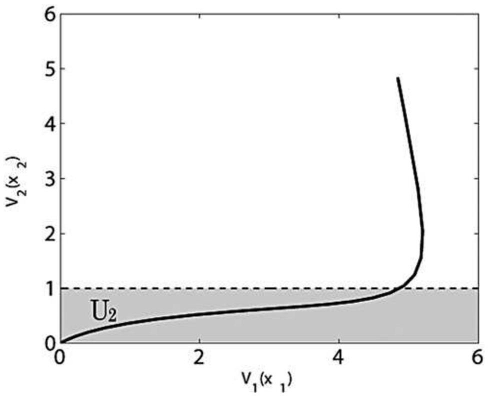

Fig. 2. The trajectory of (44)–(45) on the (V_1,V_2)-plane for $r_1(t) = 1.8/(2 + t)$ and $x(0) = [2.2,2.2]^T$ with relation to the set $\mathbf{U}_2$.

VII. CONCLUDING REMARKS

The iISS small-gain theorem developed in [6,10] for interconnections of two iISS subsystems has been revisited to give it a trajectory-based interpretation linking with the contractive mechanism of the ISS small-gain theorem. According to the preceding study, an interconnection involving a non-ISS subsystem is stable only if the other subsystem is ISS with respect to its feedback input. By making use of this fact, this paper has shown that the behavior of the interconnected system can be split into two phases. In the first phase, roughly, the trajectory of the ISS subsystem behaves almost independently of the other iISS subsystem and this phase lasts until the trajectory of the ISS subsystem enters a neighborhood $\mathbf{U}_2$ of the origin with a certain radius. In this phase, the behavior of the merely iISS subsystem is almost a free response. In the second phase, the interaction between the two subsystems takes effect and the contractive behavior of the whole state vector occurs since the small-gain constraint plays the role of the ISS small-gain condition in $\mathbf{U}_2$. This observation would be practically useful in designing and analyzing the dynamics of nonlinear control systems based on the iISS small-gain theorem. It is worth stressing that the above interpretation is not always applicable. The external signals are not allowed to be large either, as in (21)–(22) or (41)–(42). These assumptions ensure that the transient response actually dies in finite time which allows us to make use of the "ISS" small-gain argument for the subsequent behavior in dealing with "iISS" subsystems. There are interconnected systems which violate these assumptions and can anyway be proved to be iISS with respect to the external signals by constructing Lyapunov functions as in [6,10].

The independent study in [13] reported very recently also combines the transient with a small-gain argument for a system class which overlaps with the class of systems this paper deals with although the study [13] does not formulate systems in the framework of iISS. In [13], another stability property so-called input-to-output stability is verified by computing the input-to-output gain of interconnected systems under the assumption that an estimate of trajectories is somehow available during a finite time period when an ISS-type small-gain criterion is invalid. External signals can be incorporated into the stability analysis as far as the above assumption is fulfilled. In contrast, this paper does not assume anything more than the standard iISS dissipation inequalities of subsystems, which would be less demanding than the time embedded trajectory estimate used in [13]. An abstract model in [13] covers a considerably broad class of systems at the price of some complexities in the stability criterion.

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