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A new characterisation of exponential stability

Elena Panteley Antonio Loria

Abstract—We present a new characterization of exponential stability for nonlinear systems in the form of Lyapunov functions which may be upper and lower bounded by monotonic functions satisfying a growth order relationship rather than being polynomials of the state’s norm. In particular, one may allow for Lyapunov functions with arbitrary weakly homogeneous bounds.

I. INTRODUCTION

Consider the ordinary differential equation

$$\dot{x} = f(t, x) \quad (1)$$

where f is continuous in t and locally Lipschitz in x uniformly in t . Let $x = 0$ be an equilibrium point of the latter and denote the solutions with initial conditions $t_o \in \mathbb{R}_+$ and $x_o \in \mathbb{R}^n$, by $x(t, x_o, t_o)$. We recall that the trivial solution $x = 0$ is uniformly exponentially stable if there exist constants k, λ an r such that

$$|x_o| < r, t \in \mathbb{R}_+ \Rightarrow |x(t, x_o, t_o)| \leq k|x_o|e^{-\lambda(t-t_o)}. \quad (2)$$

We say that the origin is uniformly globally exponentially stable if $r = \infty$.

Exponential stability of nonlinear systems described by ordinary differential equations dates back at least to Krasovskii’s work in the late 1950s –cf. [4, Theorem 11.1]. The classical characterisation of uniform global exponential stability involves a Lyapunov function which satisfies upper and lower quadratic bounds of $|x|$. This has been extended to bounds that are polynomial of any order –cf. [2], [10], [3]:

Theorem 1 *Let $x = 0$ be an equilibrium point for $\dot{x} = f(t, x)$, where f is a locally Lipschitz function and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Then $x = 0$ is uniformly exponentially stable if and only if there exist a continuously differentiable function $V : [0, \infty) \times D \rightarrow \mathbb{R}_+$ such that*

$$k_1|x|^p \leq V(t, x) \leq k_2|x|^p, \quad (3a)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(x) \leq -k_3|x|^p \quad (3b)$$

for all $t \geq t_o \geq 0$ and all $x \in D$, where p and $k_i, i = 1, 2, 3$ are positive constants. If the assumptions hold globally, then $x = 0$ is uniformly globally exponentially stable.

Equivalent characterizations, in terms of a Lyapunov functions decreasing at sampling times have been reported in the context of adaptive control. See for instance [6], [3, Theorem 4.5] and [1]. For time-varying differential equations with

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locally Lipschitz right-hand side, the following is essentially contained in [5]:

[Integral characterization of UGES] For the dynamical system $\dot{x} = f(x, t)$ with f locally Lipschitz and $\sup_{x \neq 0} |f(x, t)|/|x| < \infty$ the origin is uniformly globally exponentially stable if and only if there exists $\gamma > 0$ and $p \geq 1$ such that

$$\int_0^\infty |x(t, x_o, t_o)|^p dt \leq \gamma|x_o|^p. \quad (4)$$

In [9] exponential stability is characterized by integral conditions for systems described by differential inclusions (convex, upper semi-continuous). Such characterization is useful to establish UGES from inputoutput interconnection properties involving, e.g. L_2 bounds. In this short note we give new *differential* characterisation of exponential stability equivalent to (3) but which does not rely on *explicit* polynomial bounds but functions satisfying a growth-order relation. Further results are established for weakly homogeneous systems.

II. MAIN RESULTS

Consider the system

$$\dot{x} = f(t, x), \quad f(t, 0) = 0 \quad (5)$$

Theorem 2 *Let $B_r := \{x \in \mathbb{R}^n : |x| < r\}$ and suppose that $f(t, \cdot)$ is Lipschitz on B_r uniformly in t . Let $V : \mathbb{R}_+ \times B_r \rightarrow \mathbb{R}_+$ be a continuously differentiable function such that for all $x \in B_r$, all $t \geq t_o$ and all $t_o \in \mathbb{R}_+$*

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \quad (6)$$

$$\dot{V}(t, x) \leq -\mu V(t, x), \quad (7)$$

where $\mu > 0$ is a constant and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ (functions $\mathbb{R}_+ \rightarrow \mathbb{R}_+$, strictly increasing, zero at zero and proper). Then, the origin of (5) is exponentially stable on B_r if and only if there exist constants $c > 0, c_1 > 0$ and $c_2 \in (0, 1)$ such that the following inequalities hold for all $s \geq 0$:

$$\alpha_1^{-1} \circ \alpha_2(s) \leq cs \quad (8)$$

$$\alpha_1^{-1} \circ (c_1 \alpha_2(s)) \leq c_2 s. \quad (9)$$

If all the conditions of the theorem are satisfied globally (i.e. if $r = +\infty$) then the origin is uniformly globally exponential.

Proof of Theorem 2

I. Sufficiency.

Following the arguments of the proof of [3, Theorem 3.8] we can show that, for any given D and any constants $\rho > 0$ and $r > 0$ which satisfy $B_r \subset D$ and $\rho < \alpha_1(r)$, all the

solutions of (5) starting from the set of initial conditions $x_0 \in \{x \in B_{\alpha_2^{-1}(\rho)}\}$ are well defined and moreover $x(x_0, t) \in B_r$ for all $t \geq 0$.

Now, let $T = \frac{1-c_1}{\mu c_1}$ if $c_1 < 1$ and $T = 1$ otherwise. From (7) it follows that $V(x(t)) \leq V(x(\tau))e^{-\mu(t-\tau)}$ for all $t \geq \tau \geq 0$ hence, $V(x(t)) \leq V(x(\tau))$ for all $\tau \in [0, t)$.

Integrating (6) from t to $t + T$ (with T defined above) we obtain

$$\begin{aligned} V(x(t+T)) - V(x(t)) &\leq -\mu \int_t^{t+T} V(x(\tau))d\tau \\ &\leq -\mu \int_t^{t+T} V(x(t+T))d\tau \\ &\leq -\mu TV(x(t+T)) \quad \forall t \geq 0 \end{aligned}$$

where in the second step above we used the fact that $V(x(\tau)) \geq V(x(t+T))$ for all $\tau \in [t, t+T]$. Hence, $(1+\mu T)V(x(t+T)) \leq V(x(t))$ for all $t \geq 0$. From this we obtain

$$\begin{aligned} V(x(t+T)) &\leq \frac{1}{1+\mu T}V(x(t)) \\ &\leq c_1V(x(t)) \quad \text{if } c_1 \leq 1 \\ V(x(t+T)) &\leq \frac{1}{1+\mu T}V(x(t)) \\ &\leq V(x(t)) \leq c_1V(x(t)) \quad \text{if } c_1 \geq 1 \end{aligned} \tag{11}$$

Therefore, $V(x(t+T)) \leq c_1V(x(t))$ for all $t \geq 0$. Using the bounds (11) and (6) we obtain

$$\alpha_1(|x(t+T)|) \leq c_1\alpha_2(|x(t)|) \quad \forall t \geq 0.$$

Then, from (9) it follows that for all $t \geq 0$

$$|x(t+T)| \leq c_2|x(t)|. \tag{12}$$

The rest follows the proof guidelines of [3, Theorem 4.5]. For any $t \geq 0$ let $N = \lfloor \frac{t}{T} \rfloor$, where $\lfloor \cdot \rfloor$ stands for the lower integer part. Divide the interval $[t - NT, t]$ into N equal subintervals then,

$$\begin{aligned} |x(t)| &\leq c_2|x(t-T)| \\ &\leq c_2^2|x(t-2T)| \\ &\vdots \\ &\leq c_2^N|x(t-NT)|. \end{aligned} \tag{13}$$

Since for all $t \geq 0$ we have $V(x(t)) \leq V(x_0)$ therefore,

$$|x(t)| \leq \alpha_1^{-1}\alpha_2(|x_0|) \quad \forall t \in [0, T] \tag{14}$$

consequently, from (8) it follows that $|x(t)| \leq c|x_0|$ for all $t \in [0, T]$. Combining the last bound with (13) we obtain

$$|x(x_0, t)| \leq c_2^N c|x_0| \leq cc_2^{t/T}|x_0| = c|x_0|e^{-bt},$$

where $b = \frac{1}{T} \ln \frac{1}{c_2}$. In case conditions (11)-(9) are satisfied globally we obtain that the system is UGES.

II. Necessity

Assume that the system (5) is (globally) exponentially stable for all $x \in D$ ($x \in R^n$) i.e., there exist $k, \gamma > 0$ such that the trajectories of (5) satisfy

$$|x(t, x_0)| \leq k|x_0|e^{-\gamma t} \quad \forall t \geq 0, x_0 \in D \quad (x_0 \in R^n).$$

Then, from the converse theorem on exponential stability (see for example [3]) it follows that there exists a Lyapunov function $V : D \rightarrow R_+$ ($V : R^n \rightarrow R_+$) that satisfies the inequalities

$$\begin{aligned} a_1|x|^2 &\leq V(x) \leq a_2|x|^2 \\ \frac{dV}{dx}f(x) &\leq -a_3|x|^2 \end{aligned}$$

for some positive constants $a_i, i = 1, 2, 3$.

Hence, the functions α_i ($i = 1, 2, 3$) in (11), (7) are given by $\alpha_i(s) = a_i s^2$. Simple calculations show that inequalities (8), (9) are satisfied. Indeed, $\alpha_1^{-1}(s) = \left(\frac{s}{a_1}\right)^{1/2}$ therefore, for arbitrary $c > 0$ we have

$$\alpha_1^{-1}(c_2\alpha_2(s)) = \left(\frac{c\alpha_2(s)}{a_1}\right)^{1/2} = \left(\frac{ca_2s^2}{a_1}\right)^{1/2} = \sqrt{\frac{ca_2}{a_1}}s.$$

From this it follows trivially that (8) is satisfied with $c = \sqrt{a_2/a_1}$ and (9) is satisfied for arbitrary $c_2 \in (0, 1)$ with $c_1 = \frac{a_1c_2^2}{a_2}$. ■

Remark 1

- If all conditions of the theorem 2 are satisfied except (8), then proceeding as before we obtain from (13) and (14) that

$$|x(t, x_0)| \leq \alpha(|x_0|)e^{-bt}$$

- The first part of the statement i.e., without requiring (8), (9) is equivalent to UGAS –cf. [8], [7].

III. EXPONENTIAL STABILITY FROM WEAKLY HOMOGENEOUS BOUNDS

Definition 1 A real function $f : R^n \rightarrow R$ is said to be homogeneous of order k if for any constant $\alpha \geq 0$ the following inequality holds:

$$f(\alpha x) = \alpha^k f(x).$$

Classical conditions imposed on the bounds of $V(x)$ and its derivative to insure exponential stability are formulated in terms of powers of x which are evidently homogeneous functions. In this section we show that this classical result can be extended to the class of systems with weakly homogeneous bounds on $V(x)$ and its derivative.

Our second theorem is stated in terms of Lyapunov functions satisfying weakly homogeneous bounds. We recall that real function $f : R_+ \rightarrow R_+$ is weakly homogeneous if for some function $L \in \mathcal{K}_\infty$ one has

$$f(\lambda x) \leq L(\lambda)f(x) \tag{15}$$

for all $\lambda > 0$.

Theorem 3 *The origin of $\dot{x} = f(x)$ is a uniformly exponentially stable equilibrium if and only if there exists continuously differentiable function $V : B_r \rightarrow \mathbb{R}_+$, a weakly homogeneous function $\alpha \in \mathcal{K}_\infty$ such that for all $x \in D$ and all $t \geq 0$*

$$a_1\alpha(|x|) \leq V(x) \leq a_2\alpha(|x|) \quad (16)$$

$$\dot{V}(x) \leq -a_3V, \quad (17)$$

for some positive constants a_1, a_2, a_3 . If all the conditions of the theorem are satisfied globally (with $r = +\infty$) then the system (5) is uniformly globally exponentially stable.

We wrap up this note with an example for which uniform global exponential stability is difficult to conclude from Theorem 1 yet, it may be concluded from our main results.

Example. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{x_2 g(x)}{1 + x_1^2} \\ \dot{x}_2 &= -x_2 + \frac{x_1 g(x)}{1 + x_2^2} \end{aligned}$$

where g is any locally Lipschitz function. It is easy to show that the origin is uniformly globally exponentially stable by invoking Theorem 3 and using the Lyapunov function

$$V(x) = \frac{1}{2}(x_1^4 + x_2^4) + x_1^2 + x_2^2.$$

Indeed the total time derivative of V yields $\dot{V} \leq -2V$. Note that this function does not have lower nor upper polynomial bounds however, the function $\alpha_1(s) := \frac{1}{2}V(s)$ is of the same growth order as $\alpha_2(s) := 2V(s)$. \diamond

Proof. I. Sufficiency.

From theorem 2 it follows that we need only to verify that conditions (8) and (9) are satisfied. To simplify the calculations let us take $a_1 = 1$ (what can be always done just by choosing e.g. $V_{new}(x) = V(x)/a_1$). Since $\alpha(s)$ is a weakly homogeneous function, then there exists $l \in \mathcal{K}_\infty$ such that $\alpha(\lambda s) \geq l(\lambda)\alpha(s)$ for all $\lambda > 0$. Then, for any $c_1 > 0$, defining $k = l^{-1}(c_1 a_2)$ we have

$$\alpha^{-1}(c_1 a_2 \alpha(s)) \leq \alpha^{-1}(l(k)\alpha(s)) \leq \alpha^{-1}(\alpha(ks)) = ks \quad (18)$$

Inequality (18) is valid for any $c_1 > 0$, therefore it is valid for $c_1 = 1$, hence (8) is satisfied. Moreover, we can always choose c_1 so that $k = l^{-1}(c_1 a_2) < 1$, so that (9) is satisfied as well. Therefore all the conditions of the theorem 2 are satisfied and therefore the system (5) is exponentially stable.

II. Necessity

The “only if” part of the proof follows directly the steps of the proof of Theorem 2. \blacksquare

IV. CONCLUSIONS

We have presented a new Lyapunov-like characterization of exponential stability for ordinary differential equations. Such characterization covers naturally sufficient and necessary conditions for particular cases such as \mathcal{K} -exponential stability and for homogeneous systems.

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