Performance analysis of MUSIC for spatially distributed sources

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***Abstract*—In this paper, the direction of arrival (DOA) localization of spatially distributed sources impinging on a sensor array is considered. The performance of the well known MUSIC estimator is studied in presence of model error due to angular dispersion of sources. Taking account of the coherently distributed source model proposed in [1], we establish closed-form expressions of the DOA estimation error and mean square error (MSE) due to both the model errors and the effects of a finite number of snapshots. We also propose closed-form expressions of the DOA estimation error as an explicit function of the model error in the special case where the theoretical covariance matrix is available. The analytical results are validated by numerical simulations and discussed in different configurations. They also allow to analysis the performance of MUSIC for coherently distributed sources.**

***Index Terms*—array signal processing, distributed sources, angular dispersion, estimation error, MSE, model error, performances, MUSIC**

I. INTRODUCTION

DOA estimation problems such as effects of model errors, the resolution of two closely spaced sources, and the geometry of antennas have been widely studied in the past, with the sources assumed to be far-field point transmitter or reflector [2] [3] [4]. Indeed, in most scenario the assumption seems to be correct, but, in some cases, physical sources, as for instance, acoustics sources [5] [6], or ocean waves [7], a spatially distributed model of the sources could be more appropriate [1] [8] [9]. In an environment of mobile communication where angular spreads can be observed in practice [10], a spatially distributed model outperforms also the point model.

The model for spatially distributed sources have been classified into two types, namely incoherently distributed (ID) sources and coherently distributed (CD) sources. On one hand, for ID sources, signals coming from any different point of the same distributed source can be considered uncorrelated, therefore the rank of the noise free correlation matrix does not equal the number of signals, as a result, many classical methods for DOA estimation of point sources can not be easily generalized to this situation. To solve this problem, many methods avoid estimating the effective dimension of pseudo signal subspace, for example, the covariance fitting methods (see for example [11] [12]). An ESPRIT-based approach [13] has been proposed to localize the ID source with less computational expense, but the problem of effective dimension of pseudo signal subspace has been ignored in this paper. On the other hand, in the scenario of CD sources, the received signal components are delayed and scaled replicas from from different points of the same signal, therefore the rank of the noise free correlation matrix equals the CD sources number, some low-complexity methods have been developed, for example, the ESPRIT-based sequential 1D searching algorithm proposed in [14] and the two-stage approach to estimate both DOA and angular spread proposed in [15], or the improved DSPE with a small angular spread distribution of a certain form (Uniform or Gaussian) proposed in [16].

While these methods promote the DOA estimation techniques for distributed sources, almost all the methods suffer from drawbacks. [11] is limited to the case of one source; [16] works only in the case of small angular spreads; most methods proposed for ID source are computationally expensive [11] [12]. Moreover, except most of covariance fitting approaches that are time consuming, and some methods that estimate both the DOA and angular spreads (eg : [14] [16]), these methods require in general that the shape of the dispersion is known which may not be true in practice. Even if it is the case, a mismatch on the angular spread dispersion between the real signals and the models will bring estimation errors on the DOAs.

In previous works, modeling mismatch has been described as random variable owing to the variation of the array element positions or the differences in element patterns and asymptotic performance has been analyzed [17] [18] [19] [20], and some calibration procedures have been proposed to improve the system performance (see for example [21], [22]). To the best of our knowledge, all these works were related to the scenario with point sources. In order to analytically derive DOA estimation error expressions in the case of distributed sources, a first approach could be to use a first order approximation as proposed in [17] [18]. However, as we will see in this paper, simulations reveal that it is not enough accurate, specially when two sources are close.

In this paper, using the model and the so called Distributed Signal Parameter Estimator (DSPE) proposed in [1], we concentrate on the CD sources and we will consider that the model error originates from the three following configurations: i) the source is assumed to be punctual; ii) the form of the angular spread distribution is known but with a bad spread dispersion; iii) the shape of the angular distribution is badly known. We first propose a new approach to obtain more accurate analytical expressions of the DOA estimation error and MSE. Some simplifications of the analytical expressions of the DOA estimation error in certain condition will be discussed. Then, the finite sample effect due to the estimation of the received data covariance matrix will be investigated. Finally, assuming that the shape of the angular distribution is known, an analytical expression of the DOA estimation error as a polynomial function of the angular spread dispersion error is proposed. This expression allows to see explicitly the influence of the model error on the DOA estimation performances, and could be useful in a future work to optimize the antenna parameters in order to reduce the DOA estimation error due to the angular spread of the sources.

The organization of this paper is as follows. The signal model and $1D$-DSPE are given in section II. In section III, the sensitivity of the estimator is theoretically analyzed. Numerical simulations are presented in section IV to validate the analytical expressions of the previous section. Finally, conclusions are given is section V.

II. Signal model and $1D$ estimator for distributed sources

Let us consider q spatially coherently distributed far-field sources impinging on an array of M sensors. The q source signals received at the array at moment t are denoted by $\mathbf{s}(t) = [s_1(t), \ldots, s_q(t)]^T$ and $\mathbf{y}(t) = [y_1(t), \ldots, y_M(t)]^T$, respectively. In the scenario of distributed sources, $\mathbf{y}(t)$ is given by:

$$\mathbf{y}(t) = \mathbf{C}(\theta)\mathbf{s}(t) + \mathbf{n}(t), \tag{1}$$

where $\mathbf{n}(t) \in \mathbb{C}^{M\times 1}$ represents additive noise, $\mathbf{C}(\theta) = [\mathbf{c}_{h_1}(\theta_1), \ldots, \mathbf{c}_{h_q}(\theta_q)] \in \mathbb{C}^{M\times q}$ is the array steering matrix composed of q steering vectors $\mathbf{c}_{hi}(\theta)$ that can be written as proposed in [1]:

$$\mathbf{c}_{hi}(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta+\phi)h_i(\phi)d\phi, \tag{2}$$

where $i = 1\ldots q$, and $\mathbf{a}(\theta)$ is the steering vector for a point source which arrives from the DOA θ. In the most general case, the steering vector $\mathbf{a}(\theta)$ is also a function of the array geometry, the sensor gains, the form of the wavefront, and other possible parameters which are supposed to be known.

The function $h(\phi)$ is introduced to describe the angular spread distribution and it can be parameterized by an angular dispersion Δ which is omitted in the notation. For instance, Uniform and Gaussian distributions which will be taken into account in section IV can be defined as:

$$h_u(\phi) = \begin{cases} \frac{1}{\Delta} & \text{if } -\frac{1}{2\Delta} < \phi < \frac{1}{2\Delta} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$h_g(\phi) = \frac{1}{\sqrt{2\pi}\Delta}\exp\left\{-\frac{\phi^2}{2\Delta^2}\right\} \tag{4}$$

where h_u stands for Uniform distribution with Δ as the function width, and h_g stands for Gaussian distribution where Δ is the standard deviation.

The source signals and the additive noise are considered to be centered gaussian independent random variables. Assuming that signals and noises are uncorrelated and the signal sources are uncorrelated with each other, the correlation matrix is given by:

$$\mathbf{R}_y = E[\mathbf{y}\mathbf{y}^H] = \mathbf{C}\mathbf{R}_s\mathbf{C}^H + \sigma_b^2\mathbf{I}, \tag{5}$$

where $E[.]$ is the expectation operator, $\mathbf{R}_s$ and σ_b^2 are the source covariance matrix and the noise variance, respectively.

Under the hypothesis that $q < M$ and $\mathbf{R}_s$ and $\mathbf{C}$ are not rank deficient, it is well known that the decomposition of $\mathbf{R}_y$ into eigenvalues λ_m and eigenvectors $\mathbf{u}_m$ is as follows :

$$\mathbf{R}_y = \sum_{m=1}^{M}\lambda_m\mathbf{u}_m\mathbf{u}_m = \mathbf{V}_s\Lambda_s\mathbf{V}_s^H + \sigma_b^2\mathbf{V}_b\mathbf{V}_b^H, \tag{6}$$

where $\mathbf{V}_s = [\mathbf{u}_1, \ldots, \mathbf{u}_q]$ spans the signal subspace defined by the columns of $\mathbf{C}$, $\mathbf{V}_b = [\mathbf{u}_{q+1}, \ldots, \mathbf{u}_M]$ spans the noise subspace defined as the orthogonal complement of $\mathbf{V}_s$, $\Lambda_s = diag\{\lambda_1, \ldots, \lambda_q\}$, and $\lambda_1 > \lambda_2 > \ldots > \lambda_q > \lambda_{q+1} = \ldots = \lambda_M = \sigma_b^2$.

The DSPE method proposed in [1] is a $2D$ MUSIC-like estimator allowing to jointly estimate the two parameters θ and Δ of the steering vector (2) when the form of the angular spread distribution is known: $\{\hat{\theta}, \hat{\Delta}\} = \frac{1}{\|\mathbf{c}^H(\theta,\Delta)\mathbf{V}_b\|^2}$. Two problems then arise. First, the form of $h(\phi)$ may not be known, second, the $2D$ search of θ and Δ may be time consuming. In the following, we will consider that the shape of $h(\phi)$ and Δ are imperfectly known and that the only parameter to estimate is θ. We will then investigate the effect of an imperfect knowledge of $h(\phi)$ and Δ, it is to say of $\mathbf{c}_h(\phi)$, on the $1D$-DSPE algorithm therefore denoted by:

$$\hat{\theta} = \arg\max_{\theta}\frac{1}{\|\mathbf{c}_h^H(\theta)\mathbf{V}_b\|^2} \tag{7}$$

III. Performance analysis

In this section, using the $1D$-DSPE estimator mentioned in section II, closed-form expressions of the estimation error and MSE are obtained first in a general case, and then we try to simplify them for special cases.

A. General case

In order to apply the $1D$-DSPE or MUSIC-like DOA estimator, the received data covariance matrix $\mathbf{R}_{y0} = E[\mathbf{y}_0\mathbf{y}_0^H]$ must be estimated from N snapshots $\{\mathbf{y}_0(t_n)\}_{n=1,\ldots,N}$ so that:

$$\hat{\mathbf{R}}_{y0} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{y}_0(t_n)\mathbf{y}_0^H(t_n). \tag{8}$$

It follows that the DOA estimator of θ_0 is :

$$\hat{\theta}_0 = \arg\max_{\theta}\frac{1}{\|\mathbf{c}_h^H(\theta)\hat{\mathbf{V}}_{b0}\|^2}. \tag{9}$$

where $\hat{\mathbf{V}}_{b0}$ is associated to the eigendecomposition of $\hat{\mathbf{R}}_{y0} = \hat{\mathbf{V}}_{s0}\hat{\Lambda}_{s0}\hat{\mathbf{V}}_{s0}^H + \hat{\mathbf{V}}_{b0}\hat{\Lambda}_{b0}\hat{\mathbf{V}}_{b0}^H$, and $\mathbf{c}_h(\theta)$ is given by (2) but with a form of the angular spread distribution $h(\phi)$ which may be different from the true distribution of the actual sources $h_{0i}(\phi)$.

Note that the parameters related to the true values are indexed by 0 to distinguish them from parameters used by the estimator which will be indexed by h. The theoretical covariance matrix of the signal related to the real steering matrix $\mathbf{C}_{h0}(\theta)$ is thus defined as $\mathbf{R}_{y0} = \mathbf{V}_{s0}\Lambda_{s0}\mathbf{V}_{s0}^H + \sigma_b^2\mathbf{V}_{b0}\mathbf{V}_{b0}^H$, and we write $\mathbf{R}_{y0} = \mathbf{R}_{yh} + \Delta\mathbf{R}_{yh}$, where $\mathbf{R}_{yh}$ would correspond to the covariance matrix of the signal related to a steering vector $\mathbf{C}_h(\theta)$ and where $\mathbf{R}_{yh} = \mathbf{V}_{sh}\Lambda_{sh}\mathbf{V}_{sh}^H + \sigma_b^2\mathbf{V}_{bh}\mathbf{V}_{bh}^H$.

We consider three cases of the model error : 1) the q sources are assumed to be punctual ($\mathbf{c}_h(\theta)=\mathbf{a}(\theta)$), it is equivalent to use the standard MUltiple Signal Classification (MUSIC); 2) h is known but with an error on Δ; 3) the shape of h is badly known. We can introduce the error on the steering vector model $\Delta\mathbf{C}=\mathbf{C}_h(\theta_0)-\mathbf{C}_{h0}(\theta_0)$. We also note $\mathbf{V}_{b0}=\mathbf{V}_{bh}+\Delta\mathbf{V}_{bh}$ where $\mathbf{V}_{bh}$ is the noise eigenmatrix which would correspond to the covariance matrix $\mathbf{R}_{yh}$ and where $\Delta\mathbf{V}_{bh}$ is an error on the noise eigenmatrix due to the model mismatch between $\mathbf{C}_h$ and $\mathbf{C}_{h0}$. Let us also introduce the error due to the finite number of snapshots N on the estimation of the data covariance matrix $\Delta\mathbf{R}_{y0}=\hat{\mathbf{R}}_{y0}-\mathbf{R}_{y0}$ and the error on the noise corresponding eigenmatrix $\Delta\mathbf{V}_{b0}=\hat{\mathbf{V}}_{b0}-\mathbf{V}_{b0}$. As $\mathbf{V}_{b0}\mathbf{V}_{b0}^H$ gives the noise subspace projector, the error due to the finite number of snapshots on the noise subspace projector is $\Delta\Pi_{b0}=\hat{\mathbf{V}}_{b0}\hat{\mathbf{V}}_{b0}^H-\mathbf{V}_{b0}\mathbf{V}_{b0}^H$. According to [3] and chapter 9 in [20], the relation of $\Delta\Pi_{b0}$ and $\Delta\mathbf{R}_{b0}$ is:

$$\Delta\Pi_{b0}=-\Pi_{b0}\Delta\mathbf{R}_{y0}\mathbf{Q}-\mathbf{Q}\Delta\mathbf{R}_{y0}\Pi_{b0}, \tag{10}$$

where $\mathbf{Q}=\mathbf{V}_{s0}(\Lambda_{s0}-\sigma_b^2\mathbf{I})^{-1}\mathbf{V}_{s0}$, $\mathbf{I}$ is an identity matrix.

According to (9), for the $i-th$ source, the DOA estimation $\hat{\theta}_i$ satisfies that the first order derivative of the denominator of (9) equals zero so that :

$$\left.\frac{\partial\mathbf{c}_h^H(\theta)\hat{\mathbf{V}}_{b0}\hat{\mathbf{V}}_{b0}^H\mathbf{c}_h^H(\theta)}{\partial\theta}\right|_{\hat{\theta}_i}=0,$$

which gives:

$$2\mathcal{R}e\{\dot{\mathbf{c}}_h^H(\hat{\theta}_i)\hat{\mathbf{V}}_{b0}\hat{\mathbf{V}}_{b0}^H\mathbf{c}_h(\hat{\theta}_i)\}=0, \tag{11}$$

where $\dot{\mathbf{c}}_h(\hat{\theta}_i)=\frac{\partial\mathbf{c}(\theta)}{\partial\theta}\mid_{\hat{\theta}_i}$.

Assuming that, $\hat{\theta}_i$ is not far from θ_i, we introduce the second order Taylor series approximations of $\mathbf{c}_h(\theta)$ and $\dot{\mathbf{c}}_h(\theta)$:

$$\mathbf{c}_h(\hat{\theta}_i)\approx\mathbf{c}_h(\theta_i)+\Delta\theta_i\dot{\mathbf{c}}_h(\theta_i)+\frac{1}{2}\Delta\theta_i^2\ddot{\mathbf{c}}_h(\theta_i), \tag{12}$$

and:

$$\dot{\mathbf{c}}_h(\hat{\theta}_i)\approx\dot{\mathbf{c}}_h(\theta_i)+\Delta\theta_i\ddot{\mathbf{c}}_h(\theta_i)+\frac{1}{2}\Delta\theta_i^2\dddot{\mathbf{c}}_h(\theta_i), \tag{13}$$

where $\Delta\theta_i=\hat{\theta}_i-\theta_i$ is the estimation error, $\ddot{\mathbf{c}}_h(\theta_i)=\frac{\partial^2\mathbf{c}(\theta)}{\partial\theta^2}\mid_{\theta_i}$, and $\dddot{\mathbf{c}}_h(\theta_i)=\frac{\partial^3\mathbf{c}(\theta)}{\partial\theta^3}\mid_{\theta_i}$.

Introducing (12) and (13) in (11), and exploiting (10) to substitute $\hat{\mathbf{V}}_{b0}\hat{\mathbf{V}}_{b0}$ yields:

$$A(\theta_i)\Delta\theta_i^2+(B_1(\theta_i)+B_2(\theta_i))\Delta\theta_i+C_1(\theta_i)+C_2(\theta_i)=0, \tag{14}$$

where the terms of order greater than 2 in $\Delta\theta_i$ have been neglected, and the scalar A,B_1,B_2,C_1,C_2 are given by:

$$\begin{aligned}A(\theta_i)=\mathcal{R}e\Big\{&\frac{1}{2}\dot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\ddot{\mathbf{c}}_h(\theta_i)+\ddot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{c}}_h(\theta_i)\\&+\frac{1}{2}\dddot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{c}_h(\theta_i)\Big\},\\B_1(\theta_i)=\mathcal{R}e&\left\{\dot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{c}}_h(\theta_i)+\ddot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{c}_h(\theta_i)\right\},\\B_2(\theta_i)=\mathcal{R}e&\left\{2\dot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\Delta\mathbf{R}_{y0}\mathbf{Q}\dot{\mathbf{c}}_h(\theta_i)\right.\\&\left.+\ddot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\Delta\mathbf{R}_{y0}\mathbf{Q}\mathbf{c}_h(\theta_i)\right\},\\C_1(\theta_i)=\mathcal{R}e&\left\{\dot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{c}_h(\theta_i)\right\},\\C_2(\theta_i)=\mathcal{R}e&\left\{\dot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\Delta\mathbf{R}_{y0}\mathbf{Q}\mathbf{c}_h(\theta_i)\right\}.\end{aligned}$$

We note $B(\theta_i)=B_1(\theta_i)+B_2(\theta_i)$ and $C(\theta_i)=C_1(\theta_i)+C_2(\theta_i)$, the expression of $\Delta\theta_i$ can be obtained by solving the 2^nd order equation (14):

$$\Delta\theta_i=\frac{-B(\theta_i)\pm\sqrt{B(\theta_i)^2-4A(\theta_i)C(\theta_i)}}{2A(\theta_i)}. \tag{15}$$

For the estimator $1D$-DSPE, it is the minimum value of the denominator that the criterion chooses to determinate the estimation result, so $\hat{\theta}_i$ should satisfy:

$$\frac{\partial^2\mathbf{c}_h^H(\theta)\hat{\mathbf{V}}_{b0}\hat{\mathbf{V}}_{b0}^H\mathbf{c}_h^H(\theta)}{\partial\theta^2}\mid_{\hat{\theta}_i}>0,$$

which allows to choose the positive one as the convenient solution:

$$\Delta\theta_i=\frac{-B(\theta_i)+\sqrt{B(\theta_i)^2-4A(\theta_i)C(\theta_i)}}{2A(\theta_i)}. \tag{16}$$

Exploiting the fact that $B_2(\theta_i)$ and $C_2(\theta_i)$ are random terms related to $\Delta\mathbf{R}_{y0}$, while $A(\theta_i),B_1(\theta_i),C_1(\theta_i)$ are deterministic, and using $E[\Delta\mathbf{R}_{y0}]=0$, the MSE can be expressed as:

$$\begin{aligned}&E[\Delta\theta_i^2]=\\&E\left[\frac{2B(\theta_i)^2-4A(\theta_i)C(\theta_i)-2B(\theta_i)\sqrt{(B(\theta_i)^2-4A(\theta_i)C(\theta_i)}}{4A(\theta_i)^2}\right]\\&=\frac{B_1(\theta_i)^2-2A(\theta_i)C_1(\theta_i)-B_1(\theta_i)\sqrt{B_1(\theta_i)^2-4A(\theta_i)C_1(\theta_i)}}{2A(\theta_i)^2}\\&+\left(\frac{1}{2A(\theta_i)^2}-\frac{B_1(\theta_i)}{2A(\theta_i)^2\sqrt{B_1(\theta_i)^2-4A(\theta_i)C_1(\theta_i)}}\right)E\left[B_2(\theta_i)^2\right]\\&+\frac{1}{A(\theta_i)\sqrt{B_1(\theta_i)^2-4A(\theta_i)C_1(\theta_i)}}E\left[B_2(\theta_i)C_2(\theta_i)\right].\end{aligned} \tag{17}$$

According to the formulas in [20], assuming a matrix $\mathbf{U}\in\mathbb{C}^{M\times M}$, $\Delta\mathbf{R}_{y0}$ satisfies:

$$E[\Delta\mathbf{R}_{y0}\mathbf{U}\Delta\mathbf{R}_{y0}^T]=\frac{1}{N}\mathbf{R}_{y0}\mathbf{U}^T\mathbf{R}_{y0}^T, \tag{18}$$

and:

$$E[\Delta\mathbf{R}_{y0}\mathbf{U}\Delta\mathbf{R}_{y0}]=\frac{1}{N}tr(\mathbf{R}_{y0}\mathbf{U}\mathbf{R}_{y0}). \tag{19}$$

Using (18) and (19), the expectations in (17) can be derived as:

$$\begin{aligned}&E\left[B_2(\theta_i)^2\right]\triangleq\varphi(\theta_i)\\&=\mathcal{R}e\left\{\dot{\mathbf{c}}(\theta_i)^H\Pi_{b0}\dot{\mathbf{c}}(\theta_i)\left[\dot{\mathbf{c}}(\theta_i)\Pi_{b0}\mathbf{Q}\mathbf{R}_{y0}\mathbf{Q}\dot{\mathbf{c}}(\theta_i)\right.\right.\\&\left.+\dot{\mathbf{c}}(\theta_i)^H\mathbf{Q}\mathbf{R}_{y0}\mathbf{Q}\dot{\mathbf{c}}(\theta_i)\right]\\&+\ddot{\mathbf{c}}(\theta_i)^H\Pi_{b0}\dot{\mathbf{c}}(\theta_i)\mathbf{c}(\theta_i)\mathbf{Q}\mathbf{R}_{y0}\mathbf{Q}\dot{\mathbf{c}}(\theta_i)\\&\left.+\ddot{\mathbf{c}}(\theta_i)^H\Pi_{b0}\ddot{\mathbf{c}}(\theta_i)\mathbf{c}(\theta_i)\mathbf{Q}\mathbf{R}_{y0}\mathbf{Q}\mathbf{c}(\theta_i)\right\},\\&E\left[B_2(\theta_i)C_2(\theta_i)\right]\triangleq\chi(\theta_i)\\&=\mathcal{R}e\left\{\dot{\mathbf{c}}(\theta_i)^H\Pi_{b0}\dot{\mathbf{c}}(\theta_i)\dot{\mathbf{c}}(\theta_i)\mathbf{Q}\mathbf{R}_{y0}\mathbf{Q}\mathbf{c}(\theta_i)\right.\\&\left.+\ddot{\mathbf{c}}(\theta_i)^H\Pi_{b0}\dot{\mathbf{c}}(\theta_i)\mathbf{c}(\theta_i)\mathbf{Q}\mathbf{R}_{y0}\mathbf{Q}\mathbf{c}(\theta_i)\right\}\end{aligned} \tag{20}$$

Finally, the MSE with two perturbations is thus denoted by (21), as shown at the bottom of the page.

It is interesting to see that the MSE is composed of two terms, one depends only on the model error, the other depends on the model error but with a factor $\sigma^2/2N$. It can be expected that the second term will be negligible when N increases.

B. First order approximation

In this subsection, we discuss the situation that the estimation error $\Delta\theta_i$ is small enough, so that the second order terms in $\Delta\theta_i$ can be negligible with respect to the first order terms. Keeping only the first order terms in (14) yields:

$$B(\theta_i)\Delta\theta_i + C(\theta_i) = 0. \tag{22}$$

Replacing $\hat{\mathbf{V}}_{b0}\hat{\mathbf{V}}_{b0}^H$ and $\mathbf{V}_{b0}$ by $\mathbf{V}_{b0}\mathbf{V}_{b0} + \Delta\Pi_{b0}$ and $\mathbf{V}_{bh} + \Delta\mathbf{V}_{bh}$, respectively, neglecting the second order terms in $\Delta\mathbf{V}_{b0}\Delta\theta_i$ and $\Delta\mathbf{V}_{bh}\Delta\theta_i$. And using $\mathbf{R}_{yh}\mathbf{V}_{b0} = \mathbf{V}_{b0}\Lambda_{bh}$ to obtain the relation of $\Delta\mathbf{V}_{bh}$ and $\Delta\mathbf{c}$, the estimation error can be simplified as:

$$\begin{aligned}\Delta\theta_i = &\frac{\mathcal{R}e\{\dot{\mathbf{c}}_h^H(\theta_0)\mathbf{V}_{bh}\mathbf{V}_{bh}^H\Delta\mathbf{c}\}}{\dot{\mathbf{c}}_h^H(\theta_0)\mathbf{V}_{bh}\mathbf{V}_{bh}^H\dot{\mathbf{c}}_h(\theta_0)} \\ &+ \frac{\mathcal{R}e\{\dot{\mathbf{c}}_h^H(\theta_0)\mathbf{V}_{bh}\mathbf{V}_{bh}^H\Delta\mathbf{R}_{y0}\mathbf{Q}\mathbf{c}_h(\theta_0)\}}{\dot{\mathbf{c}}_h^H(\theta_0)\mathbf{V}_{bh}\mathbf{V}_{bh}^H\dot{\mathbf{c}}_h(\theta_0)}.\end{aligned} \tag{23}$$

And the MSE can be given by:

$$\begin{aligned}E[\Delta\theta_i^2] = &\left(\frac{\mathcal{R}e\{\dot{\mathbf{c}}_h^H(\theta_0)\mathbf{V}_{bh}\mathbf{V}_{bh}^H\Delta\mathbf{c}\}}{\dot{\mathbf{c}}_h^H(\theta_0)\mathbf{V}_{bh}\mathbf{V}_{bh}^H\dot{\mathbf{c}}_h(\theta_0)}\right)^2 \\ &+ \frac{\sigma_b^2}{2N}\cdot\frac{\mathcal{R}e\{\dot{\mathbf{c}}_h^H(\theta_0)\mathbf{V}_{bh}\mathbf{V}_{bh}^H\dot{\mathbf{c}}_h(\theta_0)\mathbf{c}_h^H(\theta_0)\mathbf{R}_{y0}\mathbf{Q}\mathbf{c}_h(\theta_0)\}}{(\dot{\mathbf{c}}_h^H(\theta_0)\mathbf{V}_{bh}\mathbf{V}_{bh}^H\dot{\mathbf{c}}_h(\theta_0))^2}.\end{aligned} \tag{24}$$

We can notice that when the first order in $\Delta\theta_i$ is small enough, for both estimation error and the MSE, we can separate the error contributions which depends only on the model error $\Delta\mathbf{c}$(first term in (24)) and the contribution which depends on the snapshot number N(second term in (24)).

C. Estimation error as an explicit function of the model error

*1) **General case**:* As we will see later in section IV, when the snapshot number is large enough, the effect of a finite snapshot number is negligible, and the model error has a main influence on the estimation performance. In this section, it will be assumed that an exact measurement of the perturbed data covariance matrix is available. The DOA estimator becomes:

$$\hat{\theta}_i = \arg\max_{\theta}\frac{1}{\|\mathbf{c}_h^H(\theta)\mathbf{V}_{b0}\|^2}. \tag{25}$$

It follows that the estimation error $\Delta\theta_i$ is:

$$\Delta\theta_i = \frac{-B_1(\theta_i) + \sqrt{B_1(\theta_i)^2 - 4A(\theta_i)C_1(\theta_i)}}{2A(\theta_i)}. \tag{26}$$

To be able to quantify the model error, we assume that in this case, the shape of h related to the real signals sources is known, where all the sources have the same shape and the same angular spread dispersion Δ_0. The model error is therefore caused by the error on Δ or by the fact that the sources are assumed to be punctual. Assuming again that Δ is not far from Δ_0, and noting $\delta = \Delta - \Delta_0$, we can introduce the second order Taylor series approximations in δ:

$$\mathbf{c}_h(\theta_i) = \mathbf{c}_{h0}(\theta_i) + \delta\mathbf{g}_{h01}(\theta_i) + \frac{1}{2}\delta^2\mathbf{g}_{h02}(\theta_i), \tag{27}$$

where $\mathbf{g}_{h01}(\theta_i) = \frac{\partial\mathbf{c}_{h0}(\theta)}{\partial\Delta}\mid_{\Delta_0}$, $\mathbf{g}_{h02}(\theta_i) = \frac{\partial^2\mathbf{c}_{h0}(\theta)}{\partial\Delta^2}\mid_{\Delta_0}$.

We can pay attention that $\mathbf{g}_{h01}$ and $\mathbf{g}_{h02}$ reveals the sensibility of our model to the variation of the angular dispersion of the real signal.

Based on the results in appendix A, the estimation error can be thus given by:

$$\Delta\theta_i = \frac{-\Phi(\theta_i,\delta,\Delta_0) + \sqrt{\Pi(\theta_i,\delta,\Delta_0)}}{\Psi(\theta_i,\delta,\Delta_0)}, \tag{28}$$

where $\Phi(\theta_i,\delta,\Delta_0)$, $\Pi(\theta_i,\delta,\Delta_0)$ and $\Psi(\theta_i,\delta,\Delta_0)$ defined in appendix A are functions depending only on parameters θ_i,Δ_0 of signal sources and parameters of sensors, for example, the geometry of the sensor array. When we study the influence of the model error on the performance of DSPE, these parameters can be regarded as constant.

*2) **A simplified expression**:* This subsection provides a simplified explicit expression as a function of δ of the estimation error. Assuming again that we have the theoretical covariance matrix, by ignoring the second terms in $\Delta\theta_i$ in (14), the DOA estimation error is approximated as:

$$\Delta\theta_i = -\mathcal{R}e\left\{\frac{\dot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{c}_h(\theta_i)}{\dot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{c}}_h(\theta_i) + \ddot{\mathbf{c}}_h(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{c}_h(\theta_i)}\right\}. \tag{29}$$

As we hope to obtain a final expression as a polynomial as a function of δ in this case, we extend the Taylor series in δ to third order terms for a more accurate approximation:

$$\mathbf{c}_h(\theta_i) = \mathbf{c}_{h0}(\theta_i) + \delta\mathbf{g}_0(\theta_i) + \frac{1}{2}\delta^2\mathbf{g}_{02}(\theta_i) + \frac{1}{6}\delta^3\mathbf{g}_{03}(\theta_i), \tag{30}$$

where $\mathbf{g}_{03}(\theta_i) = \frac{\partial^3\mathbf{c}_{h0}(\theta)}{\partial\Delta^3}\mid_{\Delta_0}$.

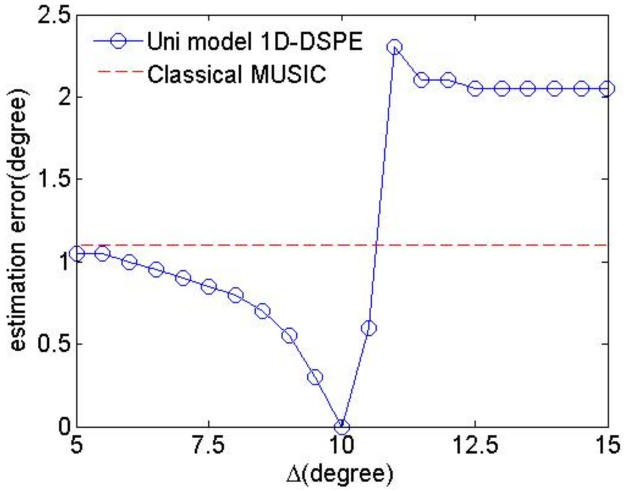
By introducing (30) in (29), results in appendix B show that the estimation error (29) can be further expressed as a polynomial in δ:

$$\Delta\theta_i = \alpha(\theta_i,\Delta_0)\delta + \beta(\theta_i,\Delta_0)\delta^2 + \gamma(\theta_i,\Delta_0)\delta^3 \tag{31}$$

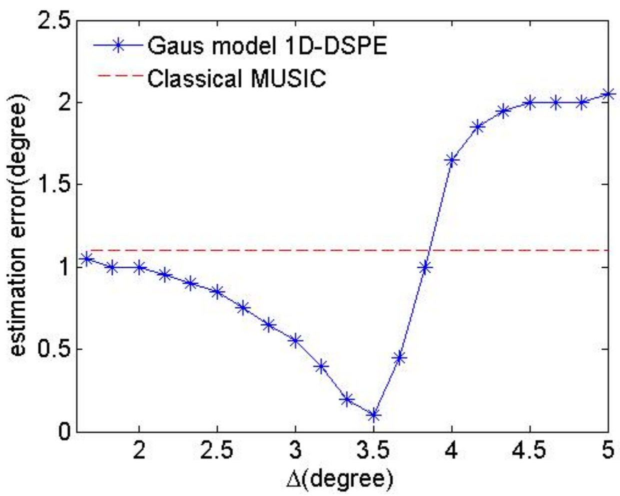
$$\begin{aligned}E[\Delta\theta_i^2] = &\frac{B_1(\theta_i)^2 - 2A(\theta_i)C_1(\theta_i) - B_1(\theta_i)\sqrt{B_1(\theta_i)^2 - 4A(\theta_i)C_1(\theta_i)}}{2A(\theta_i)^2} \\ &+ \frac{\sigma_b^2}{2N}\left[\left(\frac{1}{2A(\theta_i)^2} - \frac{B_1(\theta_i)}{2A(\theta_i)^2\sqrt{B_1(\theta_i)^2 - 4A(\theta_i)C_1(\theta_i)}}\right)\varphi(\theta_i) + \frac{\chi(\theta_i)}{A(\theta_i)\sqrt{B_1(\theta_i)^2 - 4A(\theta_i)C_1(\theta_i)}}\right].\end{aligned} \tag{21}$$

where α, β and γ are functions depending on the real signals sources and sensor parameters and defined in appendix B.

IV. Numerical simulations



(a) Uniform distributed model and classical MUSIC



(b) Gaussian distributed model and classical MUSIC

Fig. 1. Absolute DOA estimation error $|\Delta\theta_2|$ vs. angular dispersion Δ (Uniform angular dispersion, $\Delta_0 = 10°, \theta_1 = 28°, \theta_2 = 32°, M =$ 10 sensors, $SNR = 10dB$)

In this section, numerical examples are presented to illustrate the validity of the analytical results of the estimation performances established in section III. In all simulations, the uniform linear array is composed of $M = 10$ sensors spaced by $d = \lambda/2$, and $SNR = 10dB$. Different analytical results are compared to simulations results.

We start with the case where two uniform distributed signal sources arrive from $\theta_1 = 28°$ and $\theta_2 = 32°$, respectively, with an angular dispersion $\Delta_0 = 10°$, and a theoretical covariance matrix (Figure 1). The absolute values of DOA estimation error is traced as a function of different Δ in the $1D$-DSPE estimator. We can compare the performance of $1D$-DSPE and classical MUSIC in two case: the mismatch of

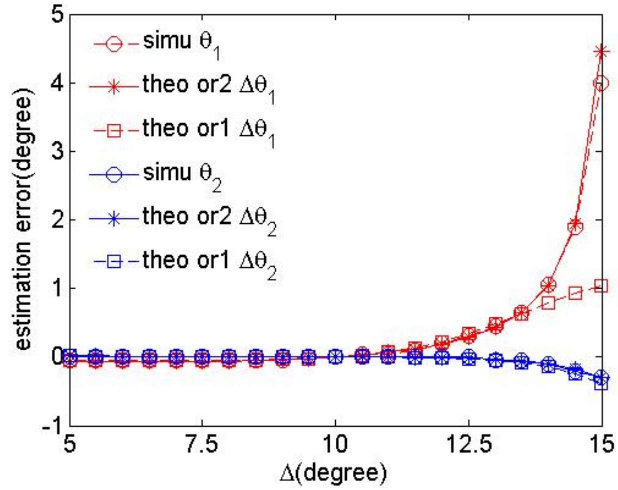
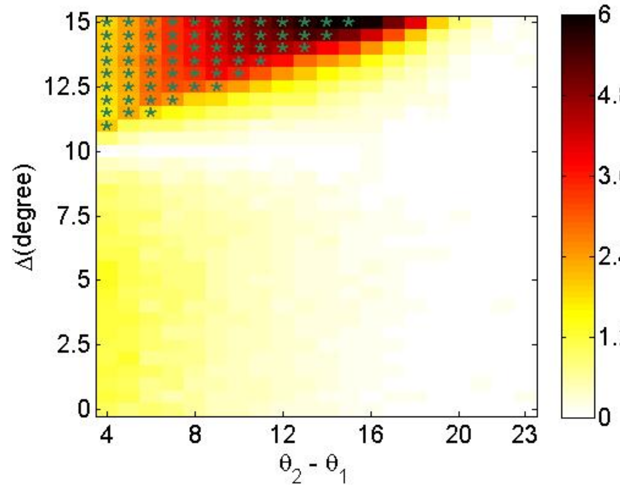


Fig. 2. DOA estimation error vs. angular dispersion Δ (Uniform angular dispersion, $\Delta_0 = 10°, \theta_1 = 21°, \theta_2 = 39°, M = 10$ sensors, $SNR = 10dB, N = 1000$)

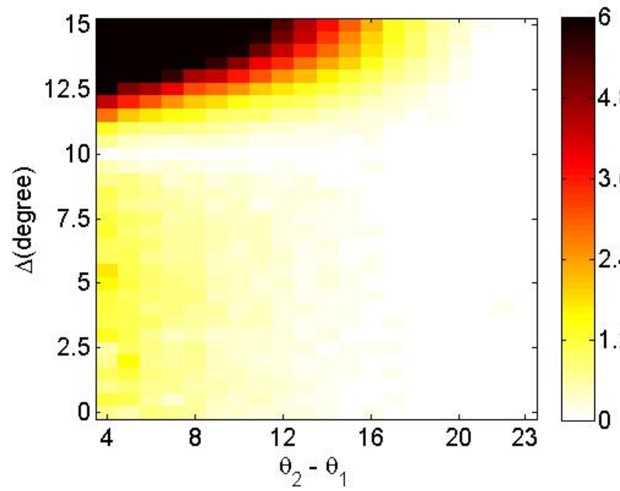
angular spread dispersion of source signal, and the mismatch of angular spread form of source signal. In Figure 1(a), we can see that when Δ_0 is over-estimated too much, the estimation error of $1D$-DSPE exceeds that of classical MUSIC. In Figure 1(b) a Gaussian model is used, attention that Δ pour Gaussian distribution presents the standard deviation, so a Δ near 3.5 corresponds to the angular distribution width of the source signal. The results reveal that even if we take a bad angular distribution form for the distributed estimator, it is possible to have a smaller DOA estimation error than classical MUSIC.

The result of test where two uniform distributed signal sources arrive from $\theta_1 = 21°$ and $\theta_2 = 39°$, respectively, with an angular dispersion $\Delta_0 = 10°$, and snapshot number is set to 1000 is illustrated in Figure 2. The model dispersion parameter Δ used by the estimator has been varied to study its effect on the DOA's estimation accuracy. We can notice that for θ_2, the DOA estimation error $\Delta\theta_2$ obtained in (16)(see : theo or2 $\Delta\theta_2$) and the one obtained in (23)(see : theo or1 $\Delta\theta_2$) both match the simulation results. However, for θ_1, which has an estimation error much bigger than θ_2 when the model error increases, the result obtained in (16)(see : theo or2 $\Delta\theta_1$) outperforms the one in (23)(see : theo or1 $\Delta\theta_1$). We can observe that for such an angular separation distance, for both the two sources, an over-estimated Δ_0 yields bigger estimation error than an under-estimated Δ_0.

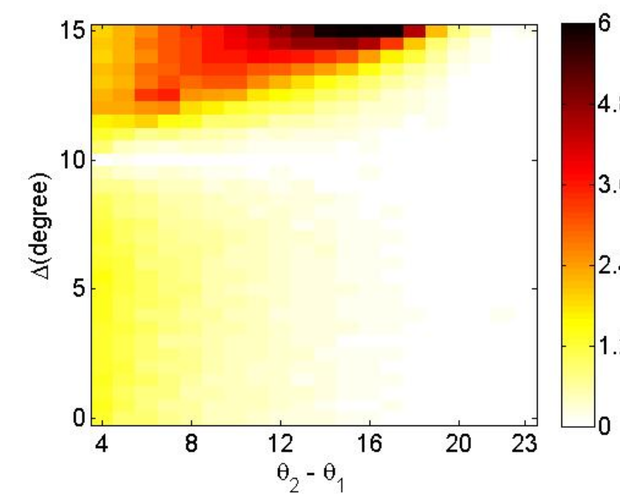
Figure 3 shows the results of the absolute value of the DOA estimation error when the model dispersion parameter and the distance between the two sources both vary, with $\theta_m = \frac{1}{2}(\theta_1 + \theta_2) = 30°$. The green stars mark the region where we have some resolution problems, that is to say, the two sources are so close that the criterion $1D$-DSPE gives the false appearance that there is only one source in the middle. In addition to the fact that the theoretical results(see : Figure 3(b) and 3(c)) correspond to the simulation result(see : Figure 3(a)), we can observe that again the result obtained in (16)(see : Figure 3(c)) works better than the result obtained in (23), even in the case



(a) Simulation

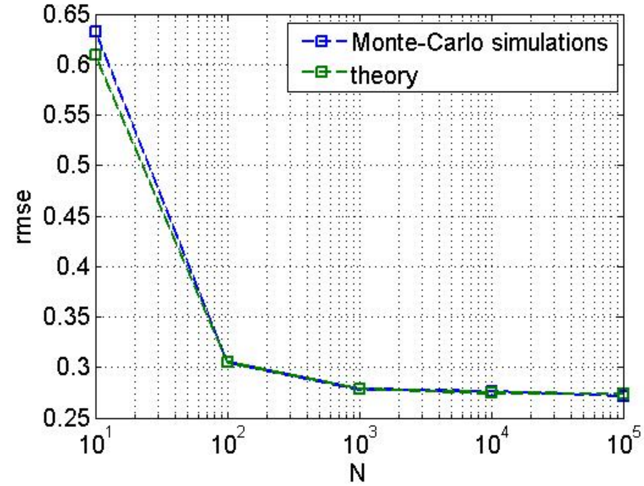


(b) Theory order 1 in $\Delta\theta_2$

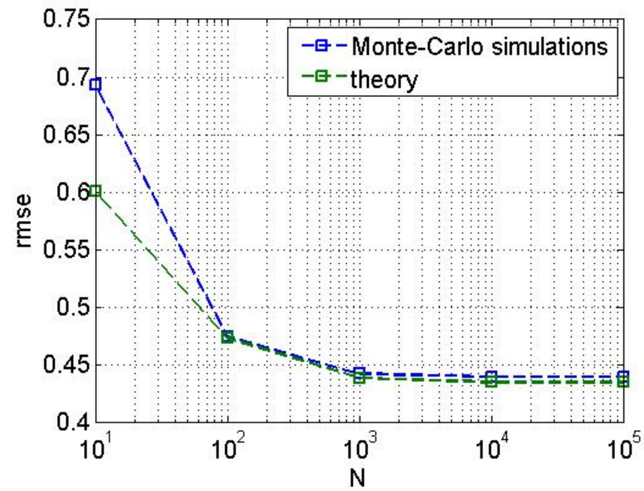


(c) Theory order 2 in $\Delta\theta_2$

Fig. 3. Absolute value of estimation error $|\Delta\theta_2|$ vs. source angular separation $|\theta_2 - \theta_1|$ and model angular dispersion Δ (Uniform angular dispersion, $\Delta_0 = 10°, \Delta = 15°, M = 10$ sensors, $SNR = 10dB$, 1000 Monte-Carlo simulations)



(a) $\theta_1(21°)$

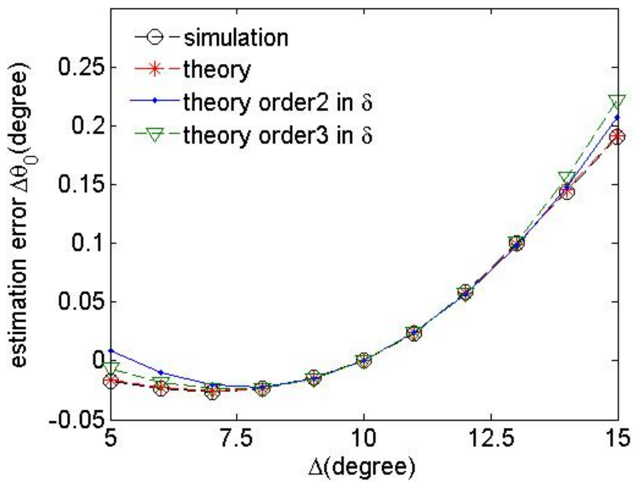


(b) $\theta_2(39°)$

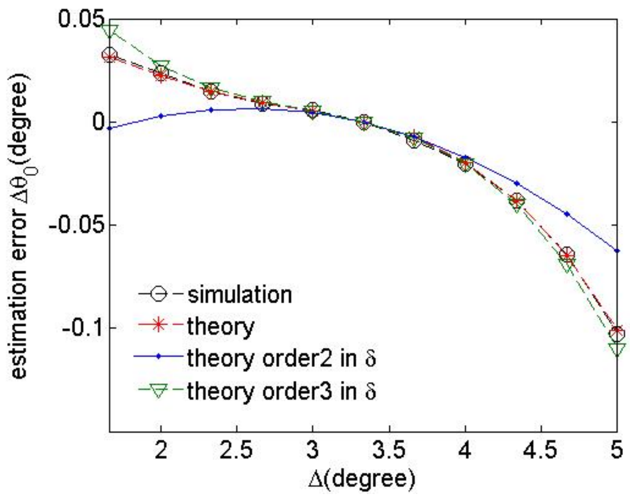
Fig. 4. Rmse vs. N (Gaussian angular dispersion, $\Delta_0 = 3.3°, \Delta = 5°, M =$ 10sensors, $SNR = 10dB$)

where there are resolution problems. It is interesting to see that with a same configuration, we can recover the estimation error in Figure 2. Taking into account that a distributed estimator with $\Delta = 0°$ means the classical estimator MUSIC, and $\Delta = 10°$ means we have a perfect 1D-DSPE, the advantage of a distributed estimator with a good distributed model to classical MUSIC is demonstrated in $3D$.

To investigate the asymptotic performance of the estimator, the RMSE of the estimator is evaluated with 100 Monte-Carlo simulations and is compared to the theoretical expression established in (21). We plot the RMSE of the estimator vs the snapshot number in Figure 4, where two Gaussian distributed sources arrive from the same DOA as in Figure 2, the angular dispersion of sources is $\Delta_0 = 3.3°$, while the model parameter is set to $\Delta = 5°$. It can be noticed that the two curves fit better when the number of samples is high. The MSE also decreases as well as N increases, and then converges to a



(a) Uniform angular dispersion ($\Delta_0 = 10°, \theta_0 = 40°$)



(b) Gaussian angular dispersion ($\Delta_0 = 3.3°, \theta_0 = 30°$)

Fig. 5. Estimation error vs. angular dispersion Δ ($M = 10$sensors$, SNR = 10dB$)

non zero value whose expression is given in (21). This reveals that when there are two perturbations, the finite number of snapshots effect dominates in the case of a small number of snapshots whereas the model error effect dominates when the snapshots number is high.

In addition, the validation of (26) (see : theory) and (31) (see : theory order3 in δ) are illustrated in Figure 5. We compare the simulation results with those obtained in (26) and in (31) and the ones obtained when we take only the first two terms of (31) (see : theory order2 in δ) in the scenario of one single source for Uniform distribution ($\theta_0 = 40\circ$, $\Delta_0 = 10°$), and Gaussian distribution($\theta_0 = 30°, \Delta_0 = 3.3°$), respectively. We can observe that the polynomial approximations in δ can give a good description for the trend of variation for the DOA estimation error as a function of the model error. In the future work, we will try to reduce the DOA estimation error by reducing the factors of δ in (31), that is to say, by optimizing the geometry or other array parameters.

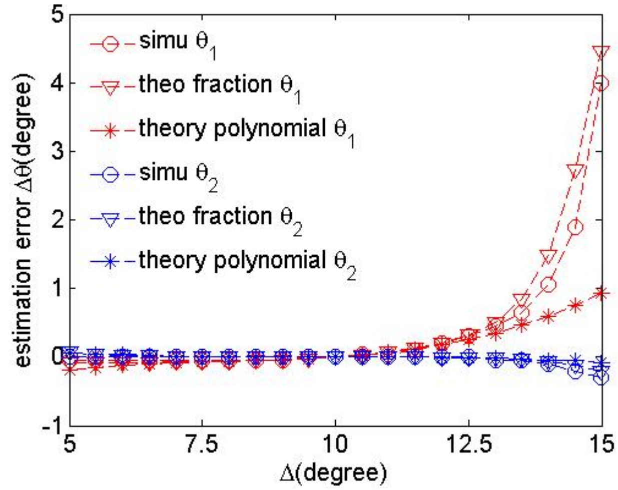


Fig. 6. DOA estimation error vs. angular dispersion model error (Uniform angular dispersion, $\Delta_0 = 10°, \theta_1 = 21°, \theta_2 = 39° M = 10$sensors$, SNR = 10dB$)

In Figure 6, we plot the estimation error approximated by (28) (see : theo fraction θ_1 and theo fraction θ_2), and approximated by (31) (see : theory polynomial θ_1 and theory polynomial θ_2) in the scenario of two Uniform distributed sources($\theta_1 = 21°, \theta_2 = 39°$). We can observe that, in this case, (28) correspond better to the simulation results, while there are too many approximations to obtain a polynomial expression (31).

V. Conclusion

In this paper, we have investigated the effects of both the angular dispersion of the source and the finite number of snapshots on the behavior of the DOA MUSIC-based estimator. New analytical expressions of the DOA estimation error and MSE as a function of these two perturbations have been given. Particularly, in the special case when the theoretical covariance matrix is available, expressions as an explicit function of the model error is proposed, which gives an easier way to analyze the influence of model error, or to optimize the array configurations to reduce the DOA estimation error. Simulations which are carried out are in adequacy with the proposed theoretical results. The performance of MUSIC for coherently distributed sources can thus be analyzed.

Appendix A

Introducing the approximation (27) in the expressions of $A(\theta_i), B_1(\theta_i)$ and $C_1(\theta_i)$ in (26), and keeping the terms in second order in $\Delta\theta_i$ yields:

$$\begin{aligned} A(\theta_i) &= f_1 + f_2\delta + f_3\delta^2, \\ B_1(\theta_i) &= f_4 + f_5\delta + f_6\delta^2, \\ C_1(\theta_i) &= f_7\delta + f_8\delta^2, \end{aligned} \tag{32}$$

where:

$$f_1 = \mathcal{R}e\left\{\frac{3}{2}\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\ddot{\mathbf{c}}_{h0}(\theta_i)\right\},$$

$$f_2 = \mathcal{R}e\left\{\frac{3}{2}\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\ddot{\mathbf{g}}_{02}(\theta_i) + \frac{3}{2}\dot{\mathbf{g}}_{02}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\ddot{\mathbf{c}}_{h0}(\theta_i) + \frac{1}{2}\dddot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\},$$

$$f_3 = \mathcal{R}e\left\{\frac{3}{4}\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\ddot{\mathbf{g}}_{02}(\theta_i) + \frac{3}{2}\dot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\ddot{\mathbf{g}}_0(\theta_i) + \frac{3}{4}\dot{\mathbf{g}}_{02}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\ddot{\mathbf{c}}_{h0}(\theta_i) + \frac{1}{4}\dddot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_{02}(\theta_i) + \frac{1}{2}\dddot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\},$$

$$f_4 = \mathcal{R}e\left\{\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{c}}_{h0}(\theta_i)\right\},$$

$$f_5 = \mathcal{R}e\left\{2\dot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{c}}_{h0}(\theta_i) + \ddot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\},$$

$$f_6 = \mathcal{R}e\left\{\frac{1}{2}\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{g}}_{02}(\theta_i) + \dot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{g}}_0(\theta_i) + \frac{1}{2}\dot{\mathbf{g}}_{02}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{c}}_{h0}(\theta_i) + \frac{1}{2}\ddot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_{02}(\theta_i) + \ddot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\},$$

$$f_7 = \mathcal{R}e\left\{\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\},$$

$$f_8 = \mathcal{R}e\left\{\frac{1}{2}\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_{02}(\theta_i) + \dot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\}.$$

Hence, the estimation error is rewritten as:

$$\Delta\theta_i = \frac{-(f_4 + f_5\delta + f_6\delta^2) + \sqrt{x + y\delta + z\delta^2 + w\delta^3 + k\delta^4}}{2(f_1 + f_2\delta + f_3\delta^2)}, \tag{33}$$

where:

$$\begin{aligned} x &= f_4^2, \\ y &= 2f_4f_5 - 4f_1f_7, \\ z &= f_5^2 + 2f_4f_6 - 4f_1f_8 - 4f_2f_7, \\ w &= 2f_5f_6 - 4f_2f_8 - 4f_3f_7, \\ k &= f_6^2 - 4f_3f_8. \end{aligned}$$

Note that $\Phi(\theta_i, \delta, \Delta_0) = f_4 + f_5\delta + f_6\delta^2$, $\Pi(\theta_i, \delta, \Delta_0) = x + y\delta + z\delta^2 + w\delta^3 + k\delta^4$ and $\Psi(\theta_i, \delta, \Delta_0) = 2(f_1 + f_2\delta + f_3\delta^2)$, the estimation error expression results in (28):

$$\Delta\theta_i = \frac{-\Phi(\theta_i, \delta, \Delta_0) + \sqrt{\Pi(\theta_i, \delta, \Delta_0)}}{\Psi(\theta_i, \delta, \Delta_0)}. \tag{34}$$

APPENDIX B

Introducing the approximation (30) in (29), and keeping the third order terms in δ, the DOA estimation error can be approximated by:

$$\Delta\theta_i = -\mathcal{R}e\left\{\frac{f_a\delta + f_b\delta^2 + f_c\delta^3}{f_d + f_e\delta + f_f\delta^2 + f_g\delta^3}\right\}, \tag{35}$$

where:

$$f_a = \mathcal{R}e\left\{\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\},$$

$$f_b = \mathcal{R}e\left\{\frac{1}{2}\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_{02}(\theta_i) + \dot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\},$$

$$f_c = \mathcal{R}e\left\{\frac{1}{2}\dot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_{02}(\theta_i) + \frac{1}{6}\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_{03}(\theta_i) + \frac{1}{2}\dot{\mathbf{g}}_{02}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\},$$

$$f_d = \mathcal{R}e\left\{\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{c}}_{h0}(\theta_i)\right\},$$

$$f_e = \mathcal{R}e\{2\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{g}}_0(\theta_i) + \ddot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\},$$

$$f_f = \mathcal{R}e\left\{\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{g}}_{02}(\theta_i) + \dot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{g}}_0(\theta_i) + \frac{1}{2}\ddot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_{02}(\theta_i) + \ddot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i)\right\},$$

$$f_g = \mathcal{R}e\left\{\frac{1}{6}\dot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{g}}_{03}(\theta_i) + \dot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\dot{\mathbf{g}}_{02}(\theta_i) + \frac{1}{2}\ddot{\mathbf{g}}_0(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_{02}(\theta_i) + \frac{1}{2}\ddot{\mathbf{g}}_{02}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_0(\theta_i) + \frac{1}{6}\ddot{\mathbf{c}}_{h0}(\theta_i)^H\mathbf{V}_{b0}\mathbf{V}_{b0}^H\mathbf{g}_{03}(\theta_i)\right\}.$$

As δ is small enough, (35) can be approximated by:

$$\begin{aligned} \Delta\theta_i \approx & -\frac{1}{f_d}(f_a\delta + f_b\delta^2 + f_c\delta^3) \\ & \cdot (1 - \frac{f_e}{f_d}\delta - (\frac{f_f}{f_d} - \frac{f_e^2}{f_d^2})\delta^2 - \frac{f_g}{f_d}\delta^3) \\ \approx & -\frac{f_a}{f_d}\delta - (\frac{f_b}{f_d} - \frac{f_af_e}{f_d^2})\delta^2 \\ & -(\frac{f_af_e^2}{f_d^3} + \frac{f_c}{f_d} - \frac{f_af_f}{f_d^2} - \frac{f_bf_e}{f_d^2})\delta^3. \end{aligned} \tag{36}$$

We note $\alpha(\theta_i, \Delta_0)$,$\beta(\theta_i, \Delta_0)$ and $\gamma(\theta_i, \Delta_0)$ for the factor of δ, δ^2 and δ^3, respectively. The expression of DOA estimation error results in (31):

$$\Delta\theta_i = \alpha(\theta_i, \Delta_0)\delta + \beta(\theta_i, \Delta_0)\delta^2 + \gamma(\theta_i, \Delta_0)\delta^3. \tag{37}$$

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