Performance analysis of MUSIC for spatially distributed sources

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***Abstract*—In this paper, the direction of arrival (DOA) localization of spatially distributed sources impinging on a sensor array is considered. The performance of the MUSIC estimator is studied in the presence of model error due to angular dispersion of sources. Taking into account the coherently distributed source model proposed in [1], we establish closed-form expressions of the DOA estimation bias and mean square error (MSE) due to both the model errors and the effects of a finite number of snapshots. The analytical results are validated by numerical simulations and discussed in different configurations. They also make possible the analysis of the performance of MUSIC for coherently distributed sources.**

***Index Terms*—array signal processing, distributed sources, angular dispersion, model error, performance, MUSIC**

I. Introduction

DOA estimation problems such as the effects of model errors, the resolution of two closely spaced sources, and the geometry of antennas have been widely studied in the past, with the sources assumed to be far-field point transmitters or reflectors [2]. Indeed, in most scenarios the assumption seems to be correct, but in some cases, as for instance the localization of acoustics [3] [4] or bio-medical [5] sources, ocean waves [6], or mobile communication [7], a spatially distributed model of the sources could be more appropriate.

The models for spatially distributed sources have been classified into two types, namely incoherently distributed (ID) sources and coherently distributed (CD) sources. On the one hand, for ID sources, signals coming from different points of the same distributed source can be considered uncorrelated, therefore the rank of the noise-free correlation matrix does not equal the number of signals [8] [9] [10]. On the other hand, in the scenario of CD sources, the received signal components are delayed and scaled replicas from different points of the same signal, therefore the rank of the noise-free correlation matrix equals the number of CD sources. A MUSIC-based estimator called Distributed Signal Parameter Estimator (DSPE) has been proposed to estimate both the DOA and the angular dispersion parameters of the sources [1]. Some low-complexity methods have been developed, for example, the ESPRIT-based sequential 1D searching algorithm proposed in [11] and the two-stage approach to estimate both DOA and angular spread proposed in [12], or the improved DSPE with a small angular spread distribution of a certain shape (Uniform or Gaussian) proposed in [13].

While these methods promote the DOA estimation techniques for distributed sources, almost all the methods suffer from drawbacks. [8] is limited to the case of one source; [13] works only in the case of small angular spreads; most methods proposed for ID sources are computationally expensive [8] [9]. Moreover, except for most covariance fitting approaches that are time consuming, and some methods that estimate both the DOA and angular spreads (eg : [11] [13]), these methods generally require that the shape of the dispersion is known, which may not be true in practice; otherwise the results in [14] show the influence of the mismatch of the shape of distribution between the signal sources and models on the DOA estimation.

In the previous works, where the modeling mismatch has been described as random variables due to the variation of the array element positions or the differences in element patterns, asymptotic performances have been analyzed [15] [16] [17] [18], and some calibration procedures have been proposed to improve the system performance (see for example [19], [20]). To the best of our knowledge, all these studies were related to the scenario with point sources. In order to analytically derive DOA estimation error expressions in the case of distributed sources, a first approach could be to use a first order approximation as proposed in [15] [16]. However, as we will see in this paper, simulations reveal that it is not accurate enough, especially when two sources are close.

We will have our attention on the CD source model proposed in [1], which is well adapted to applications such as the aero-acoustic imaging [3]. The model error can originate from the following configurations: i) the shape of the angular distribution is badly known; ii) the shape of the angular spread distribution is known but with a bad spread dispersion. We here derive accurate approximated analytical expressions of the DOA estimation bias and MSE. Assuming that the shape of the angular distribution is known, an analytical expression of the DOA estimation error as a polynomial function of the angular spread dispersion error is proposed. This expression shows explicitly the influence of the model error on the DOA estimation performance, and could be useful in a future study to optimize the antenna parameters in order to reduce the DOA estimation error due to the angular spread of the sources.

The organization of this paper is as follows. The signal model and the extended MUSIC-based estimator are given in section II. In section III, the sensitivity of the estimator is theoretically analyzed. Numerical simulations are presented in section IV to validate the analytical expressions of the previous section. Finally, conclusions are given in section V.

II. Signal model and MUSIC for distributed sources

A. Signal model

Let us consider q spatially coherently distributed far-field sources impinging on an array of M sensors. The q sources and M signals received at the array at moment t are denoted by $\mathbf{s}(t) = [s_1(t), \ldots, s_q(t)]^T$ and $\mathbf{y}(t) = [y_1(t), \ldots, y_M(t)]^T$, respectively. In the scenario of distributed sources, $\mathbf{y}(t)$ is given by:

$$\mathbf{y}(t) = \mathbf{C}(\theta)\mathbf{s}(t) + \mathbf{n}(t), \tag{1}$$

where $\mathbf{n}(t) \in \mathbb{C}^{M\times 1}$ represents additive noise, $\mathbf{C}(\theta) = [\mathbf{c}_{h_1}(\theta_1), \ldots, \mathbf{c}_{h_q}(\theta_q)] \in \mathbb{C}^{M\times q}$ is the array steering matrix composed of q steering vectors $\mathbf{c}_{hi}(\theta)$ that can be written as proposed in [1]:

$$\mathbf{c}_{hi}(\theta_i) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta_i + \phi)h_i(\phi)d\phi, \tag{2}$$

where $\mathbf{a}(\theta)$ is the steering vector for a point source which arrives from the DOA θ. In the most general case, the steering vector $\mathbf{a}(\theta)$ is also a function of the array geometry, the sensor gains, the form of the wavefront, and other possible parameters which are supposed to be known.

The function $h_i(\phi)$ is introduced to describe the angular spread distribution and it can be parameterized by an angular dispersion Δ_i which is omitted in the notation. For instance, Uniform and Gaussian distributions which will be taken into account in section IV can be defined as:

$$h^u(\phi) = \begin{cases} \frac{1}{\Delta} & \text{if} -\frac{1}{2\Delta} < \phi < \frac{1}{2\Delta} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$h^g(\phi) = \frac{1}{\sqrt{2\pi}\Delta} \exp\left\{-\frac{\phi^2}{2\Delta^2}\right\} \tag{4}$$

where h^u stands for Uniform distribution with Δ as the function width, and h^g stands for Gaussian distribution where Δ is the standard deviation.

The source signals and the additive noise are considered to be centered Gaussian independent random variables. Assuming that signals and noises are uncorrelated and the signal sources are uncorrelated with each other, the correlation matrix is given by:

$$\mathbf{R} = E[\mathbf{y}\mathbf{y}^H] = \mathbf{C}\mathbf{R}_s\mathbf{C}^H + \sigma_b^2\mathbf{I}, \tag{5}$$

where $E[.]$ is the expectation operator, $\mathbf{R}_s$ and σ_b^2 are the source covariance matrix and the noise variance, respectively.

Under the hypothesis that $q < M$ and $\mathbf{R}_s$ and $\mathbf{C}$ are not rank deficient, it is well known that the decomposition of $\mathbf{R}$ into eigenvalues λ_m and eigenvectors $\mathbf{e}_m$ is as follows :

$$\mathbf{R} = \sum_{m=1}^{M} \lambda_m \mathbf{e}_m \mathbf{e}_m = \mathbf{U}\Lambda_s\mathbf{U}^H + \sigma_b^2\mathbf{V}\mathbf{V}^H, \tag{6}$$

where $\mathbf{U} = [\mathbf{e}_1, \ldots, \mathbf{e}_q]$ spans the signal subspace defined by the columns of $\mathbf{C}$, $\mathbf{V} = [\mathbf{e}_{q+1}, \ldots, \mathbf{e}_M]$ spans the noise subspace defined as the orthogonal complement of $\mathbf{U}$, $\Lambda_s = diag\{\lambda_1, \ldots, \lambda_q\}$.

The well known method MUltiple Signal Classification (MUSIC) makes use of the orthogonal property of $\mathbf{U}$ and $\mathbf{V}$ to localize sources with the parameters of $\mathbf{C}(\theta)$ given in the model, for the $i-th$ source, the MUSIC criterion for distributed sources is denoted by:

$$\hat{\theta}_i = \underset{\theta}{\operatorname{argmax}} \frac{1}{\|\mathbf{c}_{hi}^H(\theta)\mathbf{V}\|^2}, \tag{7}$$

We can note that the algorithm is a particular case of DSPE proposed in [1], which estimates both the DOAs of sources and the dispersion parameters Δ_i in $\mathbf{c}_{hi}(\theta_i)$, where the shape of $h_i(\phi)$ is supposed to be known.

B. Model error definitions

The criterion (7) can be impacted by two types of errors. First, the actual angular spread distribution of the source h, may not be known in practice, in this case the function $\tilde{h}$ used by the estimator will be different from the actual h, that is to say, the shape of h is badly known, or the shape of h is known but with an error on Δ. Second, the covariance matrix $\mathbf{R}$ should be estimated from a finite number of snapshots, in consequence the estimated covariance matrix $\hat{\mathbf{R}}$ and its noise subspace $\hat{\mathbf{V}}$ are different from the actual $\mathbf{R}$ and $\mathbf{V}$. Taking into consideration the two types of errors, the algorithm that we study can be given as:

$$\hat{\theta}_i = \arg\max_{\theta} \frac{1}{\|\mathbf{c}_{\tilde{h}i}^H(\theta)\hat{\mathbf{V}}\|^2}. \tag{8}$$

Let us introduce the definitions related to the errors. Assuming that the estimator (8) is based on an angular distribution $\tilde{h}_i$, the model error on the steering vector is defined as $\Delta\mathbf{c}(\theta_i) = \mathbf{c}_{\tilde{h}_i}(\theta_i) - \mathbf{c}_{hi}(\theta_i)$. Similarly the model error on the covariance matrix can be defined as $\Delta\tilde{\mathbf{R}} = \tilde{\mathbf{R}} - \mathbf{R}$, where $\tilde{\mathbf{R}} = \tilde{\mathbf{C}}\tilde{\mathbf{R}}_s\tilde{\mathbf{C}}^H + \sigma_b^2\mathbf{I}$, and $\Delta\tilde{\mathbf{V}} = \tilde{\mathbf{V}} - \mathbf{V}$, where the subspace decomposition of $\tilde{\mathbf{R}}$ is given by $\tilde{\mathbf{R}} = \tilde{\mathbf{U}}\tilde{\Lambda}_s\tilde{\mathbf{U}}^H + \sigma_b^2\tilde{\mathbf{V}}\tilde{\mathbf{V}}^H$. Assuming again that the covariance matrix is estimated from N snapshots by the empirical estimator $\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{y}(t_n)\mathbf{y}^H(t_n)$, the errors are defined as $\Delta\hat{\mathbf{R}} = \hat{\mathbf{R}} - \mathbf{R}$ and $\Delta\hat{\mathbf{V}} = \hat{\mathbf{V}} - \mathbf{V}$, where $\hat{\mathbf{R}} = \hat{\mathbf{U}}\hat{\Lambda}_s\hat{\mathbf{U}}^H + \sigma_b^2\hat{\mathbf{V}}\hat{\mathbf{V}}^H$. Let us also define $\Delta\tilde{\Pi} = \tilde{\Pi} - \Pi$ and $\Delta\hat{\Pi} = \hat{\Pi} - \Pi$, where $\tilde{\Pi} = \tilde{\mathbf{V}}\tilde{\mathbf{V}}^H$, $\hat{\Pi} = \hat{\mathbf{V}}\hat{\mathbf{V}}^H$, and $\Pi = \mathbf{V}\mathbf{V}^H$.

III. Performance analysis

In this section, we will investigate the effects of an imperfect knowledge of $h(\phi)$, and the finite number of snapshots on the MUSIC algorithm.

A. General case

According to (8), for the $i-th$ source, the DOA estimation $\hat{\theta}_i$ satisfies that the first order derivative of the denominator of (8) equals zero so that :

$$\left.\frac{\partial \mathbf{c}_{\tilde{h}}^H(\theta)\hat{\Pi}\mathbf{c}_{\tilde{h}}(\theta)}{\partial\theta}\right|_{\hat{\theta}_i} = 0,$$

which gives:

$$2\mathcal{R}e\{\dot{\mathbf{c}}_{\tilde{h}}^H(\hat{\theta}_i)\hat{\Pi}\mathbf{c}_{\tilde{h}}(\hat{\theta}_i)\} = 0, \tag{9}$$

where $\dot{\mathbf{c}}_{\tilde{h}}(\hat{\theta}_i) = \frac{\partial \mathbf{c}(\theta)}{\partial\theta}\,|_{\hat{\theta}_i}$.

Assuming that, $\hat{\theta}_i$ is not far from θ_i, we introduce the second order Taylor series approximations of $\mathbf{c}_{\tilde{h}}(\hat{\theta}_i)$ and $\dot{\mathbf{c}}_{\tilde{h}}(\hat{\theta}_i)$:

$$\mathbf{c}_{\tilde{h}}(\hat{\theta}_i) \approx \mathbf{c}_{\tilde{h}}(\theta_i) + \Delta\theta_i\dot{\mathbf{c}}_{\tilde{h}}(\theta_i) + \frac{1}{2}\Delta\theta_i^2\ddot{\mathbf{c}}_{\tilde{h}}(\theta_i), \tag{10}$$

and:

$$\dot{\mathbf{c}}_{\tilde{h}}(\hat{\theta}_i) \approx \dot{\mathbf{c}}_{\tilde{h}}(\theta_i) + \Delta\theta_i \ddot{\mathbf{c}}_{\tilde{h}}(\theta_i) + \frac{1}{2}\Delta\theta_i^2 \dddot{\mathbf{c}}_{\tilde{h}}(\theta_i), \tag{11}$$

where $\Delta\theta_i = \hat{\theta}_i - \theta_i$ is the estimation error, $\ddot{\mathbf{c}}_{\tilde{h}}(\theta_i) = \frac{\partial^2 \mathbf{c}(\theta)}{\partial\theta^2} |_{\theta_i}$, and $\dddot{\mathbf{c}}_{\tilde{h}}(\theta_i) = \frac{\partial^3 \mathbf{c}(\theta)}{\partial\theta^3} |_{\theta_i}$.

In order to derive the expressions of the DOA estimation errors, we make use of the approximation of first order in [2], the relation of $\Delta\hat{\Pi}$ and $\Delta\hat{\mathbf{R}}$ can be given as:

$$\Delta\hat{\Pi} = -\Pi\Delta\hat{\mathbf{R}}\mathbf{Q} - \mathbf{Q}\Delta\hat{\mathbf{R}}\Pi, \tag{12}$$

where $\mathbf{Q} = \mathbf{U}(\Lambda_s - \sigma_b^2\mathbf{I})^{-1}\mathbf{U}^H$, $\mathbf{I}$ is the $q \times q$ identity matrix.

Introducing (10) and (11) in (9), and exploiting (12) to substitute $\hat{\Pi}$ yields:

$$A(\theta_i)\Delta\theta_i^2 + B(\theta_i)\Delta\theta_i + C(\theta_i) = 0, \tag{13}$$

where the terms of order greater than 2 in $\Delta\theta_i$ have been neglected, and the scalar A, B, C are defined bellow with $B(\theta_i) = B_1(\theta_i) + B_2(\theta_i)$ and $C(\theta_i) = C_1(\theta_i) + C_2(\theta_i)$. In the following, we omit the θ_i in the notations $A(\theta_i), B(\theta_i), C(\theta_i), B_1(\theta_i), B_2(\theta_i), C_1(\theta_i), C_2(\theta_i)$, and $\mathbf{c}(\theta_i), \dot{\mathbf{c}}(\theta_i), \ddot{\mathbf{c}}(\theta_i)$ for simplicity :

$$\begin{aligned}
A &= \mathcal{R}e\left\{\frac{1}{2}\dot{\mathbf{c}}_{\tilde{h}}^H\Pi\ddot{\mathbf{c}}_{\tilde{h}} + \ddot{\mathbf{c}}_{\tilde{h}}^H\Pi\dot{\mathbf{c}}_{\tilde{h}} + \frac{1}{2}\dddot{\mathbf{c}}_{\tilde{h}}^H\Pi\mathbf{c}_{\tilde{h}}\right\}, \\
B_1 &= \mathcal{R}e\left\{\dot{\mathbf{c}}_{\tilde{h}}^H\Pi\dot{\mathbf{c}}_{\tilde{h}} + \ddot{\mathbf{c}}_{\tilde{h}}^H\Pi\mathbf{c}_{\tilde{h}}\right\}, \\
B_2 &= \mathcal{R}e\left\{2\dot{\mathbf{c}}_{\tilde{h}}^H\Pi\Delta\hat{\mathbf{R}}\mathbf{Q}\dot{\mathbf{c}}_{\tilde{h}} + \ddot{\mathbf{c}}_{\tilde{h}}^H\Pi\Delta\hat{\mathbf{R}}\mathbf{Q}\mathbf{c}_{\tilde{h}}\right\}, \\
C_1 &= \mathcal{R}e\left\{\dot{\mathbf{c}}_{\tilde{h}}^H\Pi\mathbf{c}_{\tilde{h}}\right\}, \\
C_2 &= \mathcal{R}e\left\{\dot{\mathbf{c}}_{\tilde{h}}^H\Pi\Delta\hat{\mathbf{R}}\mathbf{Q}\mathbf{c}_{\tilde{h}}\right\}.
\end{aligned}$$

The expression of $\Delta\theta_i$ can be obtained by solving the 2^{nd} order equation (13):

$$\Delta\theta_i = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{14}$$

For the estimator MUSIC, it is the minimum value of the denominator that the criterion chooses to determinate the estimation result, so $\hat{\theta}_i$ should satisfy:

$$\frac{\partial^2 \mathbf{c}_{\tilde{h}}^H(\theta)\hat{\Pi}\mathbf{c}_{\tilde{h}}(\theta)}{\partial\theta^2} |_{\hat{\theta}_i} > 0,$$

which makes it possible to choose the positive solution of (14) as the convenient one:

$$\Delta\theta_i = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \tag{15}$$

It follows that the estimation bias is derived as:

$$E\left[\Delta\theta_i\right] = \frac{-B_1 + \sqrt{B_1^2 - 4AC_1}}{2A}, \tag{16}$$

and the MSE with two perturbations is derived as:

$$E[\Delta\theta_i^2] = \frac{B_1^2 - 2AC_1 - B_1\sqrt{B_1^2 - 4AC_1}}{2A^2} + \\ \frac{\sigma_b^2}{N}\left[\left(\frac{1}{4A^2} - \frac{B_1}{4A^2\sqrt{B_1^2 - 4AC_1}}\right)\varphi + \frac{\chi}{2A\sqrt{B_1^2 - 4AC_1}}\right], \tag{17}$$

the calculates and the definitions of φ and χ can be found in appendix A.

It is interesting to see that the MSE is composed of two terms : one depends only on the model error, the other depends on the model error but with a factor σ_b^2/N. It can be expected that the second term will be negligible when N increases.

B. First order approximation

In this subsection, we discuss the situation where the estimation error $\Delta\theta_i$ is small enough, so that the second order terms in $\Delta\theta_i$ can be negligible with respect to the first order terms. Keeping only the first order terms in (13) yields:

$$B\Delta\theta_i + C = 0. \tag{18}$$

Introducing the property $\Pi\mathbf{c}_{\tilde{h}} = \Pi\Delta\mathbf{c}$, and neglecting the second order terms in $\Delta\theta_i\Delta\mathbf{c}$ and $\Delta\theta_i\Delta\hat{\mathbf{R}}$, (18) becomes:

$$\Delta\theta_i\dot{\mathbf{c}}_{\tilde{h}}^H\Pi\dot{\mathbf{c}}_{\tilde{h}} = \mathcal{R}e\{\dot{\mathbf{c}}_{\tilde{h}}^H\Pi\Delta\mathbf{c} + \dot{\mathbf{c}}_{\tilde{h}}^H\Pi\Delta\hat{\mathbf{R}}Q\mathbf{c}_{\tilde{h}}\}. \tag{19}$$

Replacing Π by $\tilde{\Pi} - \Delta\tilde{\Pi}$, and neglecting the second order terms in $\Delta\tilde{\Pi}\Delta\theta_i, \Delta\tilde{\Pi}\Delta\mathbf{c}$ and $\Delta\tilde{\Pi}\Delta\hat{\mathbf{R}}$, the expression of the estimation error of DOA can be simplified as:

$$\Delta\theta_i = \frac{\mathcal{R}e\{\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\Delta\mathbf{c}\}}{\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\dot{\mathbf{c}}_{\tilde{h}}} + \frac{\mathcal{R}e\{\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\Delta\hat{\mathbf{R}}\mathbf{Q}\mathbf{c}_{\tilde{h}}\}}{\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\dot{\mathbf{c}}_{\tilde{h}}}. \tag{20}$$

As $\Delta\hat{\mathbf{R}}$ is of a Wishart distribution, $E[\Delta\hat{\mathbf{R}}] = 0$ [21], it follows that the estimation bias is derived as:

$$E[\Delta\theta_i] = \frac{\mathcal{R}e\{\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\Delta\mathbf{c}\}}{\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\dot{\mathbf{c}}_{\tilde{h}}}, \tag{21}$$

and the MSE can be given by:

$$E[\Delta\theta_i^2] = \left(\frac{\mathcal{R}e\{\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\Delta\mathbf{c}\}}{\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\dot{\mathbf{c}}_{\tilde{h}}}\right)^2 + \frac{\sigma_b^2}{2N}\cdot\frac{\mathcal{R}e\{\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\dot{\mathbf{c}}_{\tilde{h}}\mathbf{c}_{\tilde{h}}^H\mathbf{Q}\mathbf{R}\mathbf{Q}\mathbf{c}_{\tilde{h}}\}}{(\dot{\mathbf{c}}_{\tilde{h}}^H\tilde{\Pi}\dot{\mathbf{c}}_{\tilde{h}})^2}. \tag{22}$$

Note that this first order approximation makes it possible to express (21) and the first term of (22) as an explicit function of the model error $\Delta\mathbf{c}$.

C. Estimation error as an explicit function of the model error

*1) **General case:*** In this section, it will be assumed that an exact measurement of the perturbed data covariance matrix is available, so that $\hat{\mathbf{R}}$ is replaced by $\mathbf{R}$ and $\Delta\hat{\mathbf{R}} = 0$. The DOA estimator (8) then becomes:

$$\hat{\theta}_i = \arg\max_{\theta}\frac{1}{\|\mathbf{c}_{\tilde{h}}^H(\theta)\mathbf{V}\|^2}. \tag{23}$$

In this case B_2 and C_2 are null and (15) can be written as:

$$\Delta\theta_i = \frac{-B_1 + \sqrt{B_1^2 - 4AC_1}}{2A}. \tag{24}$$

To be able to quantify the model error, we assume that in this case, the shape of h related to the actual signal sources is known, where all the sources have the same shape and the same angular spread dispersion Δ_0. The model error is therefore caused by the error on Δ or by the fact that the sources are assumed to be punctual. Assuming again that Δ is not far from Δ_0, and noting $\delta = \Delta - \Delta_0$, we can introduce the second order Taylor series approximations in δ:

$$\mathbf{c}_{\tilde{h}}(\theta_i) \approx \mathbf{c}_h(\theta_i) + \delta \mathbf{g}_{h1}(\theta_i) + \frac{1}{2}\delta^2 \mathbf{g}_{h2}(\theta_i), \tag{25}$$

where $\mathbf{g}_{h1}(\theta_i) = \frac{\partial \mathbf{c}_h(\theta)}{\partial \Delta} \mid_{\Delta_0}$, $\mathbf{g}_{h2}(\theta_i) = \frac{\partial^2 \mathbf{c}_h(\theta)}{\partial \Delta^2} \mid_{\Delta_0}$.

We can pay attention that $\mathbf{g}_{h1}$ and $\mathbf{g}_{h2}$ reveal the sensitivity of our model to the variation of the angular dispersion of the actual signal.

Based on the results in appendix B, the estimation error can be thus given by:

$$\Delta\theta_i = \frac{-\Phi(\theta_i, \delta, \Delta_0) + \sqrt{\Upsilon(\theta_i, \delta, \Delta_0)}}{\Psi(\theta_i, \delta, \Delta_0)}, \tag{26}$$

where $\Phi(\theta_i, \delta, \Delta_0)$, $\Upsilon(\theta_i, \delta, \Delta_0)$ and $\Psi(\theta_i, \delta, \Delta_0)$ defined in appendix B are functions of δ depending only on parameters θ_i, Δ_0 of the signal sources and parameters of the sensors, as for example, the geometry of the sensor array.

*2) **A simplified expression**:* Assuming again that we have the theoretical covariance matrix, by ignoring the second terms in $\Delta\theta_i$ in (13), the DOA estimation error is approximated as:

$$\Delta\theta_i = -\mathcal{R}e \left\{ \frac{\dot{\mathbf{c}}_{\tilde{h}}^H \Pi \mathbf{c}_{\tilde{h}}}{\dot{\mathbf{c}}_{\tilde{h}}^H \Pi \dot{\mathbf{c}}_{\tilde{h}} + \ddot{\mathbf{c}}_{\tilde{h}}^H \Pi \mathbf{c}_{\tilde{h}}} \right\}. \tag{27}$$

Note that (27) is different from (20), because $\Delta\hat{\mathbf{R}} = 0$ in this scenario and we only neglect the second terms in $\Delta\theta_i^2$.

As we hope to obtain a final expression as a polynomial as a function of δ in this case, we extend the Taylor series in δ to third order terms for an accurate approximation:

$$\mathbf{c}_{\tilde{h}}(\theta_i) = \mathbf{c}_h(\theta_i) + \delta \mathbf{g}_1(\theta_i) + \frac{1}{2}\delta^2 \mathbf{g}_2(\theta_i) + \frac{1}{6}\delta^3 \mathbf{g}_3(\theta_i), \tag{28}$$

where $\mathbf{g}_3(\theta_i) = \frac{\partial^3 \mathbf{c}_h(\theta)}{\partial \Delta^3} \mid_{\Delta_0}$.

By introducing (28) in (27), results in appendix C show that the estimation error (27) can be further expressed as a polynomial in δ:

$$\Delta\theta_i = \alpha(\theta_i, \Delta_0)\delta + \beta(\theta_i, \Delta_0)\delta^2 + \gamma(\theta_i, \Delta_0)\delta^3, \tag{29}$$

where α, β and γ are functions depending on the actual signal sources and sensor parameters and defined in appendix C.

IV. Numerical simulations

In this section, numerical examples are presented to illustrate the validity of the analytical results of the estimation performances established in section III. In all simulations, an uniform linear array is composed of $M = 10$ sensors spaced by $d = \lambda/2$, where λ is the wavelength of the sources, and $SNR = 10dB$, and $N = 1000$ snapshots. Different analytical results are compared to simulation results.

The result of a test of the estimation bias where two Uniform distributed sources at $\theta_1 = 21°$ and $\theta_2 = 39°$, respectively, with a same angular dispersion $\Delta_0 = 10°$ is illustrated in Figure 1. The model error due to the dispersion parameter Δ has been varied to study its effect on the DOA's estimation accuracy. As expected, the estimation bias is null when the exact model parameter $\Delta = \Delta_0 = 10°$ is used. We can observe that the DOA estimation bias is smaller when the parameter Δ is inferior to Δ_0, than when it is superior to Δ_0. Focusing on the validity of the expressions derived in the previous section, we can notice that for both θ_1 and θ_2, the DOA estimation bias obtained in (16) outperforms the one obtained in (21), where the advantage of (16) is much more evident for θ_1, which has a bigger estimation error.

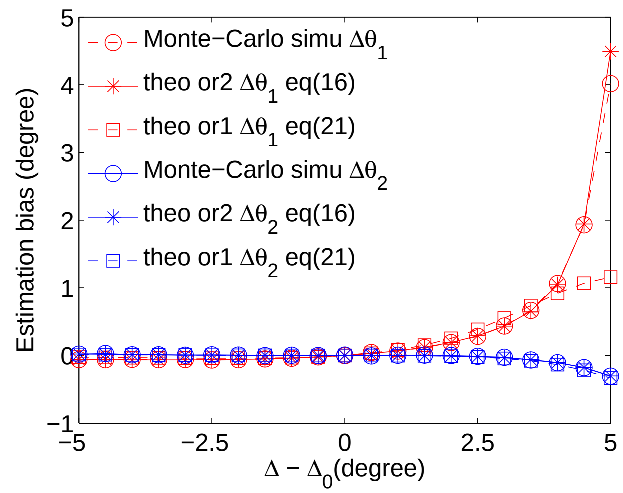
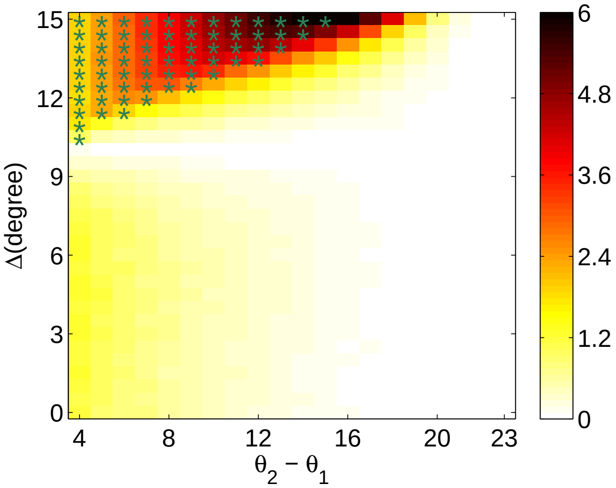


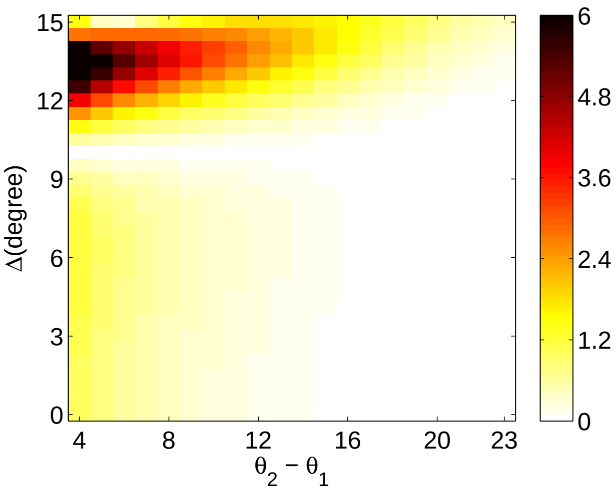
Fig. 1. The DOA estimation bias vs. the angular dispersion model error (2 sources with Uniform angular dispersion, $\Delta_0 = 10°, \theta_1 = 21°, \theta_2 = 39°$, 100 Monte-Carlo simulations)

Figure 2 shows the results of the absolute value of the DOA estimation bias when the model error due to the dispersion parameter and the angle between the two sources both vary, with $\theta_m = \frac{1}{2}(\theta_1 + \theta_2) = 30°$. The green stars mark the region where the sources are not resolved, that is to say, the two sources are so close that the MUSIC-based estimator gives the false appearance that there is only one source in the middle. These results make it possible to highlight two behaviors of the estimators. Firstly, the closer the sources are, the more the model error impacts on the estimation accuracy. Secondly, when the model error is small enough, high resolution is achieved. Taking into account that a distributed estimator with $\Delta = 0°$ means the classical estimator MUSIC with a point source model, and $\Delta = 10°$ is the extended MUSIC-based estimator without model error, the advantage of a distributed estimator with respect to a classical MUSIC is highlighted. In addition to the fact that the theoretical results correspond to the simulation results, we can observe that again the result obtained in (16) works better than the result obtained in (21), and even where there are resolution problems.

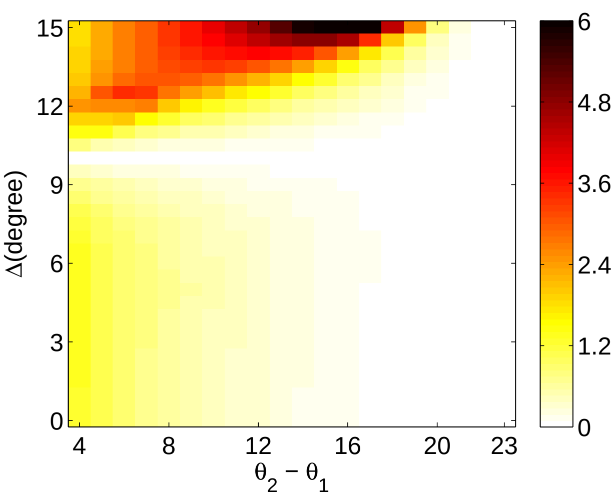
In Figure 3, the RMSE (root mean square error) of the estimator is plotted versus the snapshot number, where two Uniform distributed sources arrive from $\theta_1 = 21°$ and $\theta_2 = 39°$, the angular dispersion of the sources is $\Delta_0 = 10°$, while the model parameter is set to $\Delta = 14.5°$. The RMSE decreases as well as N increases, and then converges to a non zero value



(a) Simulation

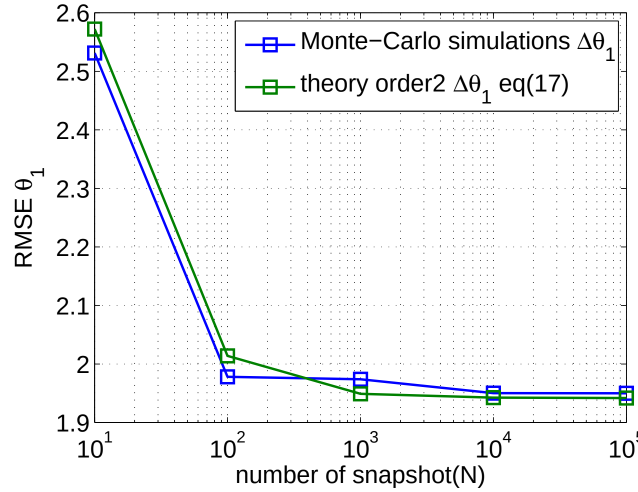


(b) Theory with order 1 approximation in (22)

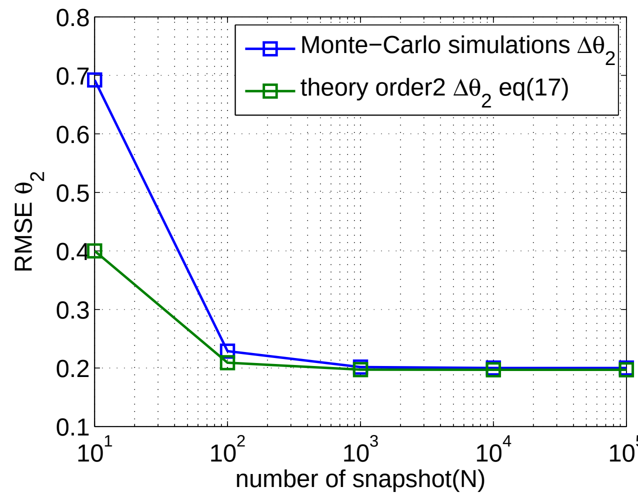


(c) Theory with order 2 approximation in (17)

Fig. 2. The absolute value of the DOA estimation bias vs. the source angular separation $|\theta_2 - \theta_1|$ and model angular dispersion Δ (2 sources with Uniform angular dispersion, $\Delta_0 = 10°, 100$ Monte-Carlo simulations)



(a) source $n°1(\theta_1 = 21°)$



(b) source $n°2(\theta_2 = 39°)$

Fig. 3. The DOA estimation RMSE vs. the number of snapshots (N) (2 sources with Uniform angular dispersion, $\Delta_0 = 10°, \Delta = 14.5°, 100$ Monte-Carlo simulations)

whose expression is given in (17). This reveals that when there are two perturbations, the finite number of snapshots effect dominates in the case of a small number of snapshots whereas the model error effect dominates when the number of snapshots is large.

Figure 4 and 5 illustrate the validation of the results obtained in (26) and (29), in order to express the estimation error as an explicit function of the model error. In the scenario of Figure 4, we have one source with Gaussian distribution ($\theta_0 = 30°, \Delta_0 = 3.3°$). Comparing simulation results with theoretical expressions obtained in (26), and with approximations on order 2 and order 3 both obtained in (29), we can observe that the polynomial expressions in δ can give a good description for the trend of variation for the DOA estimation error as a function of the model error. In Figure 5, the scenario considered is composed of two Uniform distributed sources ($\theta_1 = 21°, \theta_2 = 39°$). We can observe that, in this case,

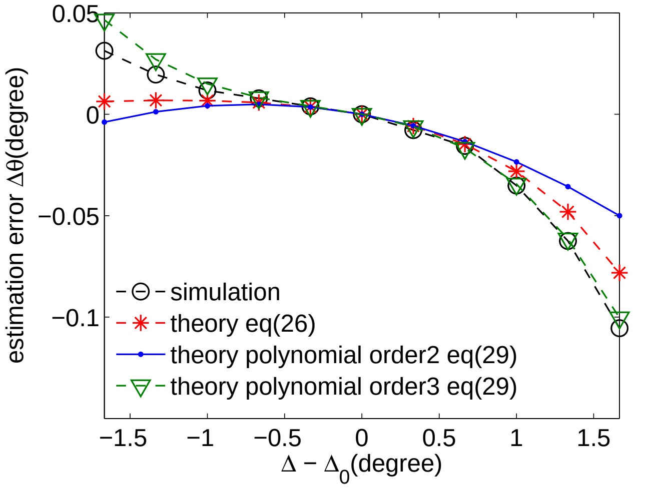


Fig. 4. The DOA estimation error vs. the angular dispersion model error (1 source with Gaussian angular dispersion, $\Delta_0 = 3.3°, \theta_0 = 30°$)

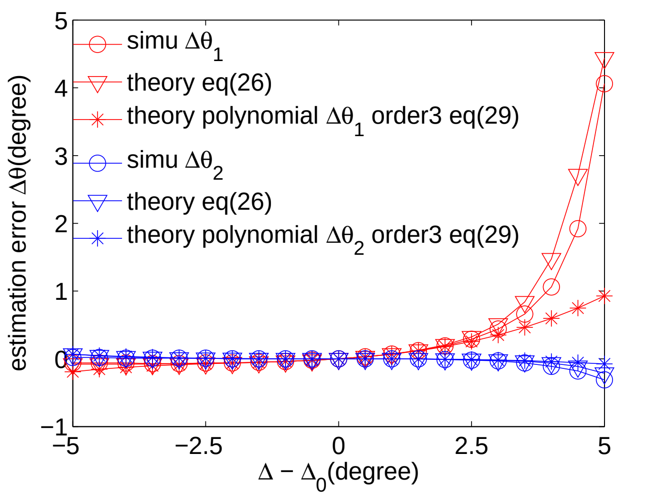
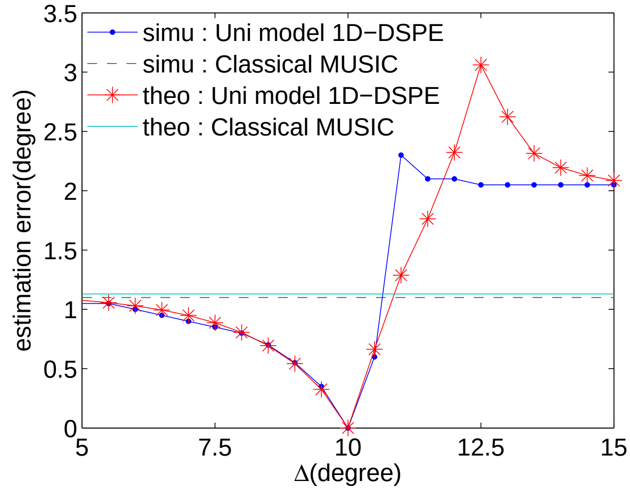


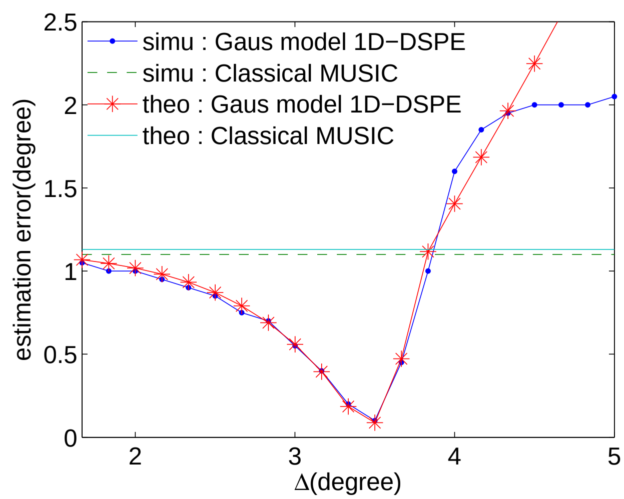
Fig. 5. The DOA estimation error vs. the angular dispersion model error (2 sources with Uniform angular dispersion, $\Delta_0 = 10°, \theta_1 = 21°, \theta_2 = 39°$)

(26) fits better the simulation results, while the polynomial expression (29) fits the simulation results only for small model error.

In Figure 6, we consider the case where two Uniform distributed signal sources arrive from $\theta_1 = 28°$ and $\theta_2 = 32°$, with an angular dispersion $\Delta_0 = 10°$, assuming that the covariance matrix is known. These results make it possible to investigate the impact of the model error due to a mismatch of the shape of the angular spread dispersion. To localize a source with Uniform distribution, we compare the performance of 3 methods: classical MUSIC, assuming a point source model (Figure 6(a),6(b)); extended MUSIC-based estimator, assuming a Uniform angular dispersion (Figure 6(a)) or a Gaussian angular dispersion (Figure 6(b)). Simulation results and theoretical results (24) are plotted against the model error due to the dispersion parameter Δ. In Figure 6(a), it is shown that the model error due to a point source model (classical



(a) The extended MUSIC-based estimator with an Uniform distributed model and the classical MUSIC estimator



(b) The extended MUSIC-based estimator with a Gaussian distributed model and the classical MUSIC estimator

Fig. 6. The absolute value of the DOA estimation error $|\Delta\theta_2|$ vs. the angular dispersion of the model (2 sources with Uniform angular dispersion, $\Delta_0 = 10°, \theta_1 = 28°, \theta_2 = 32°$)

MUSIC) provides a greater estimation error than the model error due to a mismatch on parameter Δ, except when Δ is over-estimated. When the model error is due to the use of a Gaussian shape instead of a Uniform one (Figure 6(b)), we can note that the minimum estimation error is obtained for $\Delta = 3.5°$, which is different from $\Delta_0 = 10°$, because Δ here is the standard deviation of the Gaussian model, and the minimum estimation error can not be null. The results reveal that even if we take a bad angular distribution shape which is different from the source signal for the distributed estimator, it is possible to have a smaller DOA estimation error than classical MUSIC (assuming punctual sources). Note also that in this scenario the theoretical expression fits the simulation results except when resolution problems arise (mainly when the parameter Δ is over-estimated).

V. CONCLUSION

In this paper, we have investigated the effects of both the angular dispersion of the source and the finite number of snapshots on the behavior of the MUSIC-based DOA estimator. New analytical expressions of the DOA estimation bias and MSE as a function of these two perturbations have been given. Particularly, in the special case when the theoretical covariance matrix is available, expressions as an explicit function of the model error is proposed, which gives an easier way to analyze the influence of a model error, or to optimize the array configurations to reduce the DOA estimation error. Simulations which are carried out are in adequacy with the proposed theoretical results. The performance of MUSIC for coherently distributed sources can thus be analyzed.

APPENDIX A

Based on (15), we study the case that the number of snapshots N is big enough, so that $\Delta\hat{\mathbf{R}}, B_2, C_2$ is small, we can make the approximation :

$$\sqrt{B^2-4AC} \approx \sqrt{B_1^2-4AC_1}(1+\frac{1}{2}\frac{2B_1B_2+B_2^2-4AC_2}{B_1^2-4AC_1}). \tag{30}$$

Introducing (30) in (15), and keeping only first order terms in $\Delta\hat{\mathbf{R}}$, the expression of DOA estimation error can be given by:

$$\Delta\theta_i \approx \frac{-(B_1+B_2)+\sqrt{B_1^2-4AC_1}(1+\frac{1}{2}\frac{2B_1B_2-4AC_2}{B_1^2-4AC_1})}{2A}. \tag{31}$$

$\Delta\hat{\mathbf{R}}$ is a Wishart distribution matrix with the property $E[\Delta\hat{\mathbf{R}}]=0$ according to [21], so that $E[B_2]=E[C_2]=0$. It follows that the DOA estimation bias can be derived as (16).

Similarly, the DOA estimation MSE can be given by:

$$\begin{aligned} E[\Delta\theta_i^2]=&\frac{B_1^2-2AC_1-B_1\sqrt{B_1^2-4AC_1}}{2A^2} \\ &+(1/2A^2-\frac{B_1}{2A^2\sqrt{B_1^2-4AC_1}})E[B_2^2] \\ &+\frac{1}{A\sqrt{B_1^2-4AC_1}}E[B_2C_2]. \end{aligned} \tag{32}$$

Using the same method in [15], the expressions of $E[B_2^2]$ and $E[B_2C_2]$ are given by:

$$\begin{aligned} E[B_2^2] &= \frac{\sigma_b^2}{2N}\varphi, \\ E[B_2C_2] &= \frac{\sigma_b^2}{2N}\chi, \end{aligned} \tag{33}$$

where:

$$\begin{aligned} \varphi \triangleq & \mathcal{R}e\left\{4\dot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{c}}_h(\theta_i)\dot{\mathbf{c}}_h(\theta_i)^H\mathbf{QRQ}\dot{\mathbf{c}}_h(\theta_i)\right. \\ &+4\ddot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{c}}_h(\theta_i)\mathbf{c}_h(\theta_i)\mathbf{QRQ}\mathbf{c}_h(\theta_i) \\ &\left.+\ddot{\mathbf{c}}_h(\theta_i)^H\Pi\ddot{\mathbf{c}}_h(\theta_i)\mathbf{c}_h(\theta_i)\mathbf{QRQ}\mathbf{c}_h(\theta_i)\right\}, \\ \chi \triangleq & \mathcal{R}e\left\{2\dot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{c}}_h(\theta_i)\dot{\mathbf{c}}_h(\theta_i)\mathbf{QRQ}\mathbf{c}_h(\theta_i)\right. \\ &\left.+\ddot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{c}}_h(\theta_i)\mathbf{c}_h(\theta_i)\mathbf{QRQ}\mathbf{c}_h(\theta_i)\right\}. \end{aligned}$$

Introducing (33) in (32), the DOA estimation MSE results in (17).

APPENDIX B

Introducing the approximation (25) in the expressions of $A(\theta_i), B_1(\theta_i)$ and $C_1(\theta_i)$ in (24), and keeping the terms in second order in $\Delta\theta_i$ yields:

$$\begin{aligned} A(\theta_i) &= f_1+f_2\delta+f_3\delta^2, \\ B_1(\theta_i) &= f_4+f_5\delta+f_6\delta^2, \\ C_1(\theta_i) &= f_7\delta+f_8\delta^2, \end{aligned} \tag{34}$$

where:

$$\begin{aligned} f_1 &= \mathcal{R}e\left\{\frac{3}{2}\dot{\mathbf{c}}_h(\theta_i)^H\Pi\ddot{\mathbf{c}}_h(\theta_i)\right\}, \\ f_2 &= \mathcal{R}e\left\{\frac{3}{2}\dot{\mathbf{c}}_h(\theta_i)^H\Pi\ddot{\mathbf{g}}_2(\theta_i)\right. \\ &\quad \left.+\frac{3}{2}\dot{\mathbf{g}}_2(\theta_i)^H\Pi\ddot{\mathbf{c}}_h(\theta_i)+\frac{1}{2}\dddot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\}, \\ f_3 &= \mathcal{R}e\left\{\frac{3}{4}\dot{\mathbf{c}}_h(\theta_i)^H\Pi\ddot{\mathbf{g}}_2(\theta_i)\right. \\ &\quad +\frac{3}{2}\dot{\mathbf{g}}_1(\theta_i)^H\Pi\ddot{\mathbf{g}}_1(\theta_i)+\frac{3}{4}\dot{\mathbf{g}}_2(\theta_i)^H\Pi\ddot{\mathbf{c}}_h(\theta_i) \\ &\quad \left.+\frac{1}{4}\dddot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_2(\theta_i)+\frac{1}{2}\dddot{\mathbf{g}}_1(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\}, \\ f_4 &= \mathcal{R}e\left\{\dot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{c}}_h(\theta_i)\right\}, \\ f_5 &= \mathcal{R}e\left\{2\dot{\mathbf{g}}_1(\theta_i)^H\Pi\dot{\mathbf{c}}_h(\theta_i)+\ddot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\}, \\ f_6 &= \mathcal{R}e\left\{\frac{1}{2}\dot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{g}}_2(\theta_i)\right. \\ &\quad +\dot{\mathbf{g}}_1(\theta_i)^H\Pi\dot{\mathbf{g}}_1(\theta_i)+\frac{1}{2}\dot{\mathbf{g}}_2(\theta_i)^H\Pi\dot{\mathbf{c}}_h(\theta_i) \\ &\quad \left.+\frac{1}{2}\ddot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_2(\theta_i)+\ddot{\mathbf{g}}_1(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\}, \\ f_7 &= \mathcal{R}e\left\{\dot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\}, \\ f_8 &= \mathcal{R}e\left\{\frac{1}{2}\dot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_2(\theta_i)+\dot{\mathbf{g}}_1(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\}. \end{aligned} \tag{35}$$

Hence, the estimation error is rewritten as:

$$\Delta\theta_i = \frac{-(f_4+f_5\delta+f_6\delta^2)+\sqrt{x+y\delta+z\delta^2+w\delta^3+k\delta^4}}{2(f_1+f_2\delta+f_3\delta^2)}, \tag{36}$$

where:

$$\begin{aligned} x &= f_4^2, \\ y &= 2f_4f_5-4f_1f_7, \\ z &= f_5^2+2f_4f_6-4f_1f_8-4f_2f_7, \\ w &= 2f_5f_6-4f_2f_8-4f_3f_7, \\ k &= f_6^2-4f_3f_8. \end{aligned}$$

Note that $\Phi(\theta_i,\delta,\Delta_0)=f_4+f_5\delta+f_6\delta^2$, $\Upsilon(\theta_i,\delta,\Delta_0)=x+y\delta+z\delta^2+w\delta^3+k\delta^4$ and $\Psi(\theta_i,\delta,\Delta_0)=2(f_1+f_2\delta+f_3\delta^2)$, the estimation error expression results in (26).

APPENDIX C

Introducing the approximation (28) in (27), and keeping the third order terms in δ, the DOA estimation error can be approximated by:

$$\Delta\theta_i = -\mathcal{R}e\left\{\frac{f_a\delta + f_b\delta^2 + f_c\delta^3}{f_d + f_e\delta + f_f\delta^2 + f_g\delta^3}\right\}, \tag{37}$$

where:

$$\begin{aligned}
f_a &= \mathcal{R}e\left\{\dot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\},\\
f_b &= \mathcal{R}e\left\{\frac{1}{2}\dot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_2(\theta_i) + \dot{\mathbf{g}}_1(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\},\\
f_c &= \mathcal{R}e\left\{\frac{1}{2}\dot{\mathbf{g}}_1(\theta_i)^H\Pi\mathbf{g}_2(\theta_i)\right.\\
&\quad\left.+\ \frac{1}{6}\dot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_3(\theta_i) + \frac{1}{2}\dot{\mathbf{g}}_2(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\},\\
f_d &= \mathcal{R}e\left\{\dot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{c}}_h(\theta_i)\right\},\\
f_e &= \mathcal{R}e\{2\dot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{g}}_1(\theta_i) + \ddot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\},\\
f_f &= \mathcal{R}e\left\{\dot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{g}}_2(\theta_i) + \dot{\mathbf{g}}_1(\theta_i)^H\Pi\dot{\mathbf{g}}_1(\theta_i)\right.\\
&\quad\left.+\ \frac{1}{2}\ddot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_2(\theta_i) + \ddot{\mathbf{g}}_1(\theta_i)^H\Pi\mathbf{g}_1(\theta_i)\right\},\\
f_g &= \mathcal{R}e\left\{\frac{1}{6}\dot{\mathbf{c}}_h(\theta_i)^H\Pi\dot{\mathbf{g}}_3(\theta_i)\right.\\
&\quad+\ \dot{\mathbf{g}}_1(\theta_i)^H\Pi\dot{\mathbf{g}}_2(\theta_i) + \frac{1}{2}\ddot{\mathbf{g}}_1(\theta_i)^H\Pi\mathbf{g}_2(\theta_i)\\
&\quad\left.+\ \frac{1}{2}\ddot{\mathbf{g}}_2(\theta_i)^H\Pi\mathbf{g}_1(\theta_i) + \frac{1}{6}\ddot{\mathbf{c}}_h(\theta_i)^H\Pi\mathbf{g}_3(\theta_i)\right\}.
\end{aligned}$$

As δ is small enough, (37) can be approximated by:

$$\begin{aligned}
\Delta\theta_i &\approx -\frac{1}{f_d}(f_a\delta + f_b\delta^2 + f_c\delta^3)\\
&\qquad\cdot(1 - \frac{f_e}{f_d}\delta - (\frac{f_f}{f_d} - \frac{f_e^2}{f_d^2})\delta^2 - \frac{f_g}{f_d}\delta^3)\\
&\approx -\frac{f_a}{f_d}\delta - (\frac{f_b}{f_d} - \frac{f_a f_e}{f_d^2})\delta^2\\
&\qquad - (\frac{f_a f_e^2}{f_d^3} + \frac{f_c}{f_d} - \frac{f_a f_f}{f_d^2} - \frac{f_b f_e}{f_d^2})\delta^3.
\end{aligned} \tag{38}$$

We note $\alpha(\theta_i, \Delta_0)$,$\beta(\theta_i, \Delta_0)$ and $\gamma(\theta_i, \Delta_0)$ for the factor of δ, δ^2 and δ^3, respectively. The expression of DOA estimation error results in (29).

REFERENCES

[1] S.Valaee, B.Champagne, and P.Kabal, "Parametric localization of distributed sources," *IEEE Transactions on Signal Processing*, vol. 43, no. 9, pp. 2144–2153, Sept. 1995.

[2] A.J.Weiss and B.Friedlander, "Effects of modeling errors on the resolution threshold of the MUSIC algorithm," *IEEE Transactions on Signal Processing*, vol. 42, no. 6, pp. 1519–1526, June 1994.

[3] N. Chu, A. M. Djafari, and J. Picheral, "Robust Bayesian super-resolution approach via sparsity enforcing a priori for near-field aeroacoustic source imaging," *Journal of Sound and Vibration*, vol. 332, no. 18, pp. 4369–4389, Fev. 2013.

[4] T.F.Brooks and W.M.Humphreys, "A deconvolution approach for the mapping of acoustic source(DAMAS) determined from phased microphone arrays," *Journal of Sound and Vibration*, vol. 294, no. , pp. 856–879, July 2006.

[5] G. Birot, L. Albera, F. Wendling, and I. Merlet, "Localisation of extended brain sources from EEG/MEG: the ExSo-MUSIC approach," *Elsiever Neuroimage*, vol. 56, no. 1, pp. 102–113, May 2011.

[6] A. P. Young, R. Guza, P. N. Adams, W. C. O'Reilly, and R. E. Flick, "Cross-shore decay of cliff top ground motions driven by local ocean swell and infragravity waves," *Journal of Geophysical Research: Oceans (1978–2012)*, vol. 117, no. C6, 2012.

[7] K.I.Pedersen, P.E.Mogensen, and B.H.Fleury, "A stochastic model of the temporal and azimuthal dispersion seen at the base station in outdoor propagation environments," *IEEE Transaction on Vehicular Technology*, vol. 49, no. 2, Mar. 2000.

[8] O.Besson and P.Stoica, "Decoupled estimation of DOA and angular spread of a spatially distributed source," *IEEE Transaction on Signal Processing*, vol. 48, no. 7, pp. 1872–1882, July 2000.

[9] A.Zoubir, Y.Wang, and P. Chargé, "Efficient subspace-based estimator for localization of multiple incoherently distributed sources," *IEEE Transaction on Signal Processing*, vol. 56, no. 2, pp. 532–542, Feb. 2008.

[10] A.Hu, T.Lv, H.Gao, Z.Zhang, and S.Yang, "An ESPRIT-based approach for 2-d localization of incoherently distibuted sources in massive MIMO SYSTEMS," *IEEE Journal of Selected Topics in Signal processing*, 2014.

[11] J.Lee, I.Song, H.Kwon, and S.Lee, "Low-complexity estimation of 2D DOA for coherently distributed sources," *Signal Processing*, vol. 83, no. 8, pp. 1789–1802, 2003.

[12] M.Souden, "A two-stage approach to estimate the angles of arrival and the angular spreads of locally scattered sources," *IEEE Transaction on Signal Processing*, vol. 56, no. 5, pp. 1968–1983, May 2008.

[13] A.Zoubir and Y.Wang, "Efficient DSPE algorithm for estimating the angular parameters of coherently distributed sources," *Signal Processing*, vol. 88, pp. 1071–1078, 2007.

[14] J. A. Chaaya, J. Picheral, and S. Marcos, "Localization of spatially distributed near-field sources with unknown angular spread shape," *Signal Processing*, vol. 106, pp. 259–265, 2015.

[15] A.Swindlehurst and T.Kailath, "A performance analysis of subspace method in the presence of model errors : Case of MUSIC," *IEEE Transactions on Signal Processing*, vol. 40, no. 7, pp. 1758–1773, July 1992.

[16] A.Kangas, P.Stoica, and T.Soderstrom, "Finite sample and modeling error effects on ESPRIT and MUSIC diretion estimators," *Radar, Sonar and Navigation, IEE Proceedings*, vol. 141, no. 5, pp. 249–255, Oct. 1994.

[17] A. Ferréol, P. Larzabal, and M. Viberg, "Statistical analysis of the MUSIC algorithm in the presence of modeling errors, taking into account the resolution probability," *IEEE Transactions on Signal Processing*, vol. 58, no. 5, pp. 4156–4166, Aug. 2010.

[18] S. Marcos, *Les méthodes à haute résolution, traitement d'antenne et analyse spectrale*, H. S. Publications, Ed. 8,quai du March-Neuf,75004,Paris: Editions HERMES, 1998, no. ISSN 1159-103X.

[19] Z.-M. LIU, "Conditional Cramér-Rao lower bounds for DOA estimation and array calibration," *Signal Processing Letters*, vol. 21, no. 3, pp. 361–364, Mar. 2014.

[20] H. Mir, "A generalized transfer-function based array calibration technique for direction finding," *IEEE transaction on Signal Processing*, vol. 56, no. 2, pp. 851–855, Feb. 2008.

[21] J. A. Tague and C. I. Caldwell, "Expectations of useful complex wishart forms," *Multidimensional Systems and Signal Processing*, vol. 5, no. 3, pp. 263–279, 1994.