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Abstract: In control related studies, convex liftings have been of use to solve inverse parametric linear/quadratic programming problem. This paper presents a so-called *convex liftings based method* for robust control design of constrained linear systems affected by bounded additive disturbances. It will be shown that a geometrical construction as convex lifting can be used in optimization-based control design to guarantee robust stability and recursive feasibility in a given controllable region of the state space. Finally, a numerical example will be considered to illustrate this method.

1. INTRODUCTION

Robust control plays an important role in control theory. In particular, for constrained discrete-time linear systems, robust control design in the presence of bounded additive disturbances and/or polytopic uncertainty, has been of interest in countless studies. Different design techniques have been put forward as in Kothare et al. [1996], Scokaert and Mayne [1998], Mayne et al. [2005], Rakovic et al. [2012], Bemporad et al. [2003], Grancharova and Johansen [2012], Gutman and Cwikel [1987], Blanchini [1994, 1995], Nguyen [2014], etc.

Linear matrix inequality (LMI) has been early applied in model predictive control (MPC), in Kothare et al. [1996] to design robust controller in the presence of polytopic model uncertainties. This method requires at each sampling time solving a computationally demanding LMI problem. Subsequently, based on dynamic programming, min-max optimization based method in Scokaert and Mayne [1998] has solved an MPC problem for discrete-time, linear invariant systems subject to bounded, additive disturbances. This method aims to minimize at each sampling instant, the worst case of cost function, subject to exponentially increasing set of constraints once the prediction horizon increases. Later, it is shown that a robust linear MPC problem can be alternatively solved via parametric convex programming to design explicit robust control in the presence of bounded additive disturbances and polytopic model uncertainties, see e.g. Bemporad et al. [2003]. Also, approximation of robust explicit control laws for nonlinear MPC in the presence disturbances has been studied in Grancharova and Johansen [2012]. On the other hand, tube based MPC has been originated in Mayne et al. [2005] and developed in Rakovic et al. [2012], providing new insight in robust control design. Another line of robust control was originated in Gutman and Cwikel [1987] based on *positively invariant sets* and has bloomed via different studies e.g. Blanchini [1994], Nguyen [2014] showing their simple formulations and easy implementations.

In the same line with the last studies, this paper introduces another approach based on *convex liftings* which can serve as Lyapunov functions. This method will be proved to guarantee the recursive feasibility and closed loop stability. In terms of implementation, it only requires solving a simple linear programming problem at each sampling instant.

Convex liftings have been used in studies related to structural properties of parametric convex programming based control laws. To our best knowledge, the present approach is the first attempt to use convex lifting as a direct design method.

Notation

Throughout this paper, $\mathbb{N}, \mathbb{N}_{>0}, \mathbb{R}, \mathbb{R}_+$ denote the set of non-negative integers, the set of strictly positive integers, the set of real numbers and the set of non-negative real numbers, respectively. For ease of presentation, with a given $N \in \mathbb{N}_{>0}$, by $\mathcal{I}_N$, we denote the index set: $\mathcal{I}_N = \{i \in \mathbb{N}_{>0} \mid i \leq N\}$.

A polyhedron is the intersection of finitely many halfspaces. A polytope is a bounded polyhedron. If P is an arbitrary polytope, then by $\mathcal{V}(P)$, we denote the set of its vertices. If $\mathcal{S}$ is a finite set, then $\mathrm{conv}(\mathcal{S})$ denotes the convex hull of $\mathcal{S}$. Also, for a given set $\mathcal{S}$, by $\mathrm{int}(\mathcal{S})$, we denote the interior of $\mathcal{S}$. Further, we use $\dim(\mathcal{S})$ to denote the dimension of its affine hull.

Given a set $\mathcal{S} \subset \mathbb{R}^d$ and a matrix $A \in \mathbb{R}^{d\times d}$, then $A\mathcal{S}$ is defined as follows: $A\mathcal{S} = \{As \mid s \in \mathcal{S}\}$.

Given two sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^d$, their Minkowski sum is denoted by $\mathcal{S}_1 \oplus \mathcal{S}_2$ and is defined by:

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = \left\{x \in \mathbb{R}^d \mid \exists y_1 \in \mathcal{S}_1, y_2 \in \mathcal{S}_2 \text{ s.t. } x = y_1 + y_2\right\}.$$

Also, $\mathcal{S}_1 \backslash \mathcal{S}_2$ is defined as follows:

$$\mathcal{S}_1 \backslash \mathcal{S}_2 := \left\{x \in \mathbb{R}^d \mid x \in \mathcal{S}_1,\ x \notin \mathcal{S}_2\right\}.$$

Further, given two different points $x, y \in \mathbb{R}^d$, we use $\rho(x, y)$ to denote the Euclidean distance between x and y. If $y = 0$, this distance is briefly written by $|x|$. Moreover, given a set $\mathcal{A} \subset \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, we denote $\rho_{\mathcal{A}}(x) = \inf_{y\in\mathcal{A}} \rho(x, y)$. It is clear that if $x \in \mathcal{A}$, then $\rho_{\mathcal{A}}(x) = 0$. The distance from a point to a set is also known as the Hausdorff distance and can be understood as a particular case of the distance between two sets.

This paper is organized in five sections. The problem statement is presented in Section 2. Our main results will be introduced in Section 3. An illustrative example will be considered in Section 4. The final section summarizes the contribution of the present paper.

2. PROBLEM SETTING

In this paper, we concentrate on the class of discrete-time linear invariant systems, affected by bounded additive disturbances:

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where x_k, u_k denote the state, control variables at time k, w_k stands for the disturbance at time k. The state, control variables and the disturbances are subject to constraints:

$$x_k \in \mathbb{X} \subset \mathbb{R}^{d_x}, u_k \in \mathbb{U} \subset \mathbb{R}^{d_u}, w_k \in \mathbb{W} \subset \mathbb{R}^{d_x}, \tag{2}$$

where $d_x, d_u \in \mathbb{N}_{>0}$, $\mathbb{X}, \mathbb{U}, \mathbb{W}$ are polytopes. It is assumed that $\mathbb{X}, \mathbb{U}, \mathbb{W}$ contain the origin in their interior.

The aim is to find a state feedback control law which exhibits robustness with respect to additive disturbances such that the closed loop is robustly stable. It is clear that if disturbance w_k is unknown for the computation of control action at instant k, one cannot expect to be able to guarantee the asymptotic stability of the origin. The asymptotic stability is replaced with an ultimate boundedness notion Khalil [2002], Kofman et al. [2007]. The following classical assumption is necessary for the existence of stabilizing control laws.

Assumption 2.1. The pair (A, B) is stabilizable and full-state measurement is available for control.

3. CONVEX LIFTINGS BASED CONTROL DESIGN

3.1 Disturbance invariant sets with respect to a stabilizing control law

Positively invariant sets have been studied over three decades. Due to their relevance in control theory, they turn out to be of help in many control related studies e.g. Bitsoris [1988], Blanchini and Miani [2007], Rakovic et al. [2012], Nguyen [2014],

In particular, disturbance invariant sets are meaningful in robust control design for system (1). Some remarkable results on the structure, properties and algorithms for positively invariant sets can be found in Kolmanovsky and Gilbert [1998], Rakovic et al. [2005, 2004].

The definition of a positively invariant set for linear system (1) is recalled below.

Definition 3.1. Given the dynamic system (1) subject to constraints (2), with respect to Assumption 2.1, a set Ω is called *positively invariant* with a linear control law $u_k = Kx_k \in \mathbb{U}$ if and only if $(A + BK)\Omega \oplus \mathbb{W} \subseteq \Omega$.

Such an Ω defined in Definition 3.1 is alternatively called *disturbance invariant set*. Algorithms for approximating maximal and minimal disturbance invariant sets can be found in Kolmanovsky and Gilbert [1995, 1998], Rakovic et al. [2005], Gilbert and Tan [1991]. It will be considered in the developments, presented next, that such approximations are available for control design.

Also, for the linear system (1) satisfying Assumption 2.1, it is easy to find a linear stabilizing state feedback $u_k = Kx_k$ via the solution of the Riccati equation with a pre-chosen, positive semidefinite weighting matrices, Q, R. The influence of disturbances can be taken into account in the design of unconstrained stabilizing linear feedback Boyd et al. [1994].

Note that in the presence of persistent disturbances, Ω is considered as a full-dimensional set. Otherwise, if system (1) is not affected by additive disturbances and/or is subject to polytopic model uncertainties, $\Omega = \{0\}$ can also be chosen. However, these cases are beyond the scope of this paper.

3.2 Domain of attraction

A domain of attraction is known to be a subset of all points which can be driven to a target set. To guarantee the convergence to a disturbance invariant set Ω, a domain of attraction denoted by $\mathcal{X}$, should ensure that for any point belonging to $\mathcal{X}$, there always exists control law satisfying constraint (2), which steers the state to Ω. The following definition of a contractive set, inherited from Definition 2.5 in Blanchini [1994], is of help for our development.

Definition 3.2. Given λ, $0 \leq \lambda \leq 1$, a set S is called $\lambda-$*contractive* if for any $x \in S \subseteq \mathbb{X}$, there exists $u(x) \in \mathbb{U}$ such that $(Ax + Bu(x)) \oplus \mathbb{W} \subseteq \lambda S$. If $\lambda = 1$, S is said *control invariant*.

According to Blanchini [1994], the maximal $\lambda-$contractive set, denoted by P_λ, is defined as the set containing all $\lambda-$contractive sets for system (1) subject to constraint (2). A computation of this set is recalled as follows.

$$\begin{aligned} &S_1 = \mathbb{X}, \\ &S_{i+1} = \{x \in \mathbb{X} \mid \exists u(x) \in \mathbb{U}, \text{ s.t.} \\ &\qquad\qquad (Ax + Bu(x)) \oplus \mathbb{W} \subseteq \lambda S_i\}, \text{ for } i \in \mathbb{N}_{>0}, \\ &P_\lambda = S_\infty. \end{aligned}$$

Details about algorithms for computation of P_λ can be found in Blanchini [1994], Kerrigan [2001]. For our development, we will use the maximal $\lambda-$contractive set P_λ for $0 \leq \lambda < 1$, as a domain of attraction; i.e. $\mathcal{X} = P_\lambda$.

3.3 Convex lifting construction

Convex lifting is in principle a purely geometrical notion. In control theory, the optimal cost function to a parametric linear programming problem, known as a convex lifting, is used to facilitate the implementation of explicit control laws, see e.g. Baotic et al. [2008], Jones et al. [2006]. Subsequently, it has been of use to solve inverse parametric linear/quadratic programming problem in Nguyen et al. [2014b,a, 2015]. It is worth stressing that the term "convex function" deployed in Hempel et al. [2013, 2015] completely differs from a convex lifting defined here. Before recalling its definition, some additional notation will be introduced.

Definition 3.3. A collection of $N \in \mathbb{N}_{>0}$ full-dimensional polyhedra denoted as $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$, is called a *polyhedral partition of a polyhedron* $\mathcal{X} \subseteq \mathbb{R}^{d_x}$ if:

- $\bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i = \mathcal{X}$.
- $\text{int}(\mathcal{X}_i) \bigcap \text{int}(\mathcal{X}_j) = \emptyset$ with $i \neq j$, $(i, j) \in \mathcal{I}_N^2$,

Also, $(\mathcal{X}_i, \mathcal{X}_j)$ are called neighbours if $(i, j) \in \mathcal{I}_N^2$, $i \neq j$ and $\dim(\mathcal{X}_i \cap \mathcal{X}_j) = d_x - 1$. Also, if $\mathcal{X}$ is a polytope, then $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ is called *a polytopic partition*.

Definition 3.4. For a given polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^{d_x}$, a *piecewise affine lifting* is described by the function $z : \mathcal{X} \to \mathbb{R}$ with:

$$z(x) = a_i^T x + b_i \text{ for any } x \in \mathcal{X}_i, \tag{3}$$

and $a_i \in \mathbb{R}^{d_x}$, $b_i \in \mathbb{R}$, $\forall i \in \mathcal{I}_N$.

Definition 3.5. Given a polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^{d_x}$, a piecewise affine lifting $z(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$, is called *convex lifting* if the following conditions hold true:

- $z(x)$ is continuous over $\mathcal{X}$,
- for each $i \in \mathcal{I}_N$, $z(x) > a_j^T x + b_j$ for all $x \in \mathcal{X}_i \backslash \mathcal{X}_j$ and all $j \neq i$, $j \in \mathcal{I}_N$.

It is clear that a given polyhedral partition has to satisfy some conditions for the existence of a convex lifting. Interested readers can find a summary of existence conditions in Rybnikov [2000], Nguyen et al. [2014b].

We present now an algorithm for the construction of a convex lifting which will be of use later in the proposed control law design as a Lyapunov function. This convex lifting denoted as $\ell(x)$, is defined over a domain of attraction $\mathcal{X}$. Recall that in this paper, as discussed in Section 3.2, we choose the maximal $\lambda-$contractive region P_λ, for a given $0 \leq \lambda < 1$ as a domain of attraction.

Algorithm 1 Construct a convex lifting

Input: A given positively invariant set $\Omega \subset \mathbb{R}^{d_x}$, the domain of attraction $\mathcal{X} = P_\lambda \subset \mathbb{R}^{d_x}$ with a given $0 \leq \lambda < 1$ and a scalar $a > 0$.

Output: A convex lifting $\ell(x)$ such that $\ell(x) = 0$ for every $x \in \Omega$.

1: $V_1 = \mathcal{V}(\Omega)$, $\widehat{V}_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in V_1 \right\} \subset \mathbb{R}^{d_x+1}$.

2: $V_2 = \mathcal{V}(\mathcal{X})$, $\widehat{V}_2 = \left\{ \begin{bmatrix} x \\ a \end{bmatrix} \mid x \in V_2 \right\} \subset \mathbb{R}^{d_x+1}$.

3: $\Pi = \mathrm{conv}(\widehat{V}_1 \bigcup \widehat{V}_2)$.

4: Solve the parametric linear programming problem:
$$z^*(x) = \min_z z \quad \text{s.t.} \quad \begin{bmatrix} x^T & z \end{bmatrix}^T \in \Pi. \tag{4}$$

5: $\ell(x) = z^*(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$.

Steps 1-2 in Algorithm 1 aim to lift the vertices of $\Omega, \mathcal{X}$ to $\mathbb{R}^{d_x+1}$ with appropriate heights. Namely, the vertices of Ω are lifted with heights equal to 0, whereas the vertices of $\mathcal{X}$ are lifted with heights equal to the given $a > 0$. Also the convex lifting $\ell(x)$ is generated from the parametric linear programming problem (4). The following observation describes the properties of such an $\ell(x)$, generated from Algorithm 1.

Lemma 3.6. The function $\ell(x)$ over $\mathcal{X}$, generated from Algorithm 1, is continuous, non-negative and convex. Also, $\ell(x) = 0$ for every $x \in \Omega$ and $\ell(x) > 0$ for any $x \in \mathcal{X}\backslash\Omega$.

Proof. The continuity and convexity of $\ell(x)$ can be easily derived from Theorem IV.3 in Gal [1995].

The second statement is deduced from the construction in step 1. Indeed, consider $x \in \Omega$, then x can be written as a convex combination of the vertices of Ω as: $x = \sum_{v\in V_1} \alpha(v)v$ with $\alpha(v) \geq 0$ and $\sum_{v\in V_1} \alpha(v) = 1$. It is known that $\ell(x)$ over Ω is an affine function, then $\ell(x) = a_i^T x + b_i$ leads to $\ell(x) = 0$ for every $x \in \Omega$.

To complete the proof, we need to show that $\ell(x)$ is a non-negative function. Indeed, as shown above, $\ell(x) = a_i^T x + b_i = 0$ for every $x \in \Omega$, then due to the full dimension of Ω, $a_i = 0$, $b_i = 0$. By the definition of a convex lifting, $\ell(x)$ is a piecewise affine function, thus over a region $\mathcal{X}_j$, one has $\ell(x) = a_j^T x + b_j$ for every $x \in \mathcal{X}_j$. This satisfies the convexity condition for $\mathcal{X}_j \neq \Omega$ ($\mathcal{X}_i = \Omega$):

$$a_j^T x + b_j > a_i^T x + b_i = 0, \quad \text{for every} \quad x \in \mathcal{X}_j \backslash \mathcal{X}_i,$$
$$a_j^T x + b_j = a_i^T x + b_i = 0, \quad \text{for every} \quad x \in \mathcal{X}_j \cap \mathcal{X}_i.$$

The same inclusion for the other affine functions of $\ell(x)$, leads to the non-negativity of $\ell(x)$. Moreover, $\ell(x) > 0$ for every $x \in \mathcal{X}\backslash\Omega$. The proof is complete. □

A simple consequence of the above lemma can be deduced as follows.

Lemma 3.7. For any $x \in \mathcal{X}$ and $0 \leq \beta \leq 1$, $\ell(\beta x) \leq \beta \ell(x)$.

Proof. Due to the convexity of $\ell(x)$ over $\mathcal{X}$ as proved in Lemma 3.6, we obtain
$$\ell(\beta x + (1-\beta)0) \leq \beta \ell(x) + (1-\beta)\ell(0).$$
Due to the assumption that $0 \in \mathrm{int}(\mathbb{W})$, then $0 \in \mathrm{int}(\Omega)$, meaning that $\ell(0) = 0$. This inclusion and the above one imply that $\ell(\beta x) \leq \beta \ell(x)$. □

3.4 Robust control law design procedure

The present subsection introduces the procedure for designing robust control laws based on convex liftings. Accordingly, some definitions of stability are recalled. They will be of use in the context of stability guaranteed by the proposed procedure. First, the fundamental definition of stability for nominal dynamics plays an important role for the extension of stability in the presence of disturbances.

Definition 3.8. For a discrete-time autonomous system $x_{k+1} = f(x_k)$, $x_k \in \mathbb{R}^{d_x}$, $d_x \in \mathbb{N}_{>0}$, where $f : \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}$ is a locally Lipschitz function and $f(0) = 0$, the origin is called *stable* in the sense of Lyapunov if for any $\epsilon > 0$ there exists a $\delta > 0$ such that:
$$|x_0| \leq \delta \quad \Rightarrow \quad |x_k| \leq \epsilon \quad \forall k \geq 0$$
Also, the origin is called *asymptotically stable* if it is *stable* and $\lim_{k\to\infty} |x_k| = 0$.

Based on the above definition, a definition of robust stability for a discrete time-invariant system in the presence of bounded disturbances
$$x_{k+1} = f(x_k, w_k), \quad f : \mathcal{X} \times \mathbb{W} \to \mathcal{X}, \tag{5}$$
is presented below with respect to a positively invariant set denoted by Ω.

Definition 3.9. System (5) is called *robustly stable* with respect to a positively invariant set Ω and a domain of attraction $\mathcal{X}$ if for any $\epsilon > 0$ and $x_0 \in \mathcal{X}\backslash\Omega$, there exists a $T \in \mathbb{N}_{>0}$ satisfying $\rho_\Omega(x_k) \leq \epsilon$ for every $k \geq T$, $w_t \in \mathbb{W}$, $t \geq 0$.

Many design methods rely on suitable Lyapunov functions to guarantee closed loop stability. Robust stability in the sense of Lyapunov, is recalled below for the particular case of linear systems.

Definition 3.10. Given a positively invariant set Ω, let us consider the linear system (1) subject to constraints (2) and a control law $u = \kappa(x) \in \mathbb{U}$. The closed loop is called *robust stable* if there exists a Lyapunov function $V(x) : \mathcal{X} \to \mathbb{R}_+$ and an $\alpha \in [0, 1)$ such that:
$$V(Ax + B\kappa(x) + w) - \alpha V(x) \leq 0, \tag{6}$$
for all $w \in \mathbb{W}$ and $x \in \mathcal{X}\backslash\Omega$.

We now present a procedure based on convex lifting, constructed in Algorithm 1, for robust control design. This procedure is summarized via Algorithm 2.

Algorithm 2 Robust control design procedure based on convex liftings

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Input: A convex lifting $\ell(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$, $i \in \mathcal{I}_N$ as in Algorithm 1. A positively invariant set Ω associated with a stabilizing control law $u = Kx$ over Ω.

Output: Control law $u^*(x_k)$ at each sampling time.

1: Compute $\ell(x_k)$.

2: **If** $x_k \in \Omega$ then $u^*(x_k) = Kx_k$, jump to Step 6.

3: **Else** Solve the following linear programming problem:

$$\begin{aligned} \left[\alpha^* \ (u_k^*)^T\right]^T &= \arg \min_{\left[\alpha \ u_k^T\right]^T} \alpha \\ \text{s.t. } & a_i^T(Ax_k + Bu_k + w_k) + b_i \leq \alpha \ell(x_k) \\ & \alpha \geq 0, u_k \in \mathbb{U}, \ \forall i \in \mathcal{I}_N, \forall w_k \in \mathcal{V}(\mathbb{W}). \end{aligned} \tag{7}$$

4: Apply $u^*(x_k) = u_k^*$

5: **End**

6: $k \leftarrow k + 1$, return to Step 1.

Remark 3.11. Note that the task of verifying whether or not x_k belongs to Ω in Step 2, can be easily carried out by checking whether or not $\ell(x_k) = 0$. This property is due to the construction of a convex lifting in Algorithm 1. Therefore, it is not necessary to store the constraints describing Ω in the implementation.

Natural questions arise here whether or not the linear programming problem (7) is feasible and whether closed loop stability is guaranteed by the proposed procedure. These questions are answered via the following theorem. Accordingly, we show that convex lifting constructed in Algorithm 1 can serve as a Lyapunov function. Thus the proposed control design can guarantee the robust stability as per Definition 3.10.

Theorem 3.12. Given a positively invariant set Ω associated with a robust stabilizing control law gain K, and a domain of attraction $\mathcal{X} = P_\lambda$ for a given $0 \leq \lambda < 1$, if $x_k \in \mathcal{X}$, then the linear programming problem (7) is recursively feasible. Furthermore, the closed loop is robustly stable in the sense of Lyapunov.

Proof. As for the feasibility of (7), one can easily see that $0 \leq \ell(x) \leq a$ by the construction in Algorithm 1. Therefore, due to the contractivity of $\mathcal{X}$, for any $x_k \in \mathcal{X}$ there always exists $u^*(x_k) \in \mathbb{U}$ such that: $Ax_k + Bu^*(x_k) + w_k \in \lambda\mathcal{X} \subset \mathcal{X}$ for all $w_k \in \mathbb{W}$, therefore

$$\ell(Ax_k + Bu^*(x_k) + w_k) \leq a, \text{ for every } w_k \in \mathbb{W}.$$

Due to this boundedness, the recursive feasibility of the linear programming problem (7) is ensured for a large enough gain α at each sampling time.

As for robust stability, we will prove that

$$\ell(Ax_k + Bu^*(x_k) + w_k) < \ell(x_k) \text{ for every } w_k \in \mathbb{W}.$$

Indeed, due to the contractivity of $\mathcal{X}$, for any $v \in \mathcal{V}(\mathcal{X})$, there exists a control law, denoted by $u(v) \in \mathbb{U}$ such that $Av + Bu(v) + w_k \in \lambda\mathcal{X}$ despite the disturbances $w_k \in \mathbb{W}$. For each $w_k \in \mathbb{W}$, there exists $y(w_k) \in \mathcal{X}$ such that $Av + Bu(v) + w_k = \lambda y(w_k)$. Due to Lemma 3.7, this inclusion leads to

$$\ell(Av + Bu(v) + w_k) = \ell(\lambda y(w_k)) \leq \lambda \ell(y(w_k)) \tag{8}$$

Due to the construction of $\ell(x)$ in Algorithm 1, we obtain

$$\ell(y(w_k)) \leq a. \tag{9}$$

Also, according to Algorithm 1,

$$\ell(v) = a. \tag{10}$$

From (8), (9), (10), we can deduce that

$$\ell(Av + Bu(v) + w_k) \leq \lambda \ell(v). \tag{11}$$

Note that (11) holds for every $w_k \in \mathbb{W}$. Moreover, it can be observed that:

$$\ell(Av+Bu^*(v)+w_k) \leq \ell(Av+Bu(v)+w_k), \forall w_k \in \mathbb{W}. \tag{12}$$

(11), (12) lead to the following fact:

$$\ell(Av + Bu^*(v) + w_k) \leq \lambda \ell(v), \ \forall w_k \in \mathbb{W}. \tag{13}$$

Note that (13) holds true for any vertex of $\mathcal{X}$. Now, consider a point $x_k \in \mathcal{X}_i$ in the polytopic partition $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ of $\mathcal{X}$ over which $\ell(x)$ is defined. Without loss of generality, suppose $\mathcal{X}_i \neq \Omega$, then x_k can be described via a convex combination of the vertices of $\mathcal{X}_i$, meaning:

$$x_k = \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) v, \text{ where } \alpha(v) \in \mathbb{R}_+, \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) = 1.$$

Recall that due to the definition of convex lifting, $\ell(x)$ over $\mathcal{X}_i$ is an affine function, then $\ell(x_k)$ can be written in the following form:

$$\ell(x_k) = \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) \ell(v). \tag{14}$$

If $v \in \mathcal{V}(\mathcal{X}_i)$ is a vertex of Ω, then due to the positive invariance of Ω with respect to a linear feedback $u^*(x) = Kx$, it satisfies

$$\ell(v) = 0 = \ell((A + BK)v + w_k) \text{ for every } w_k \in \mathbb{W}. \tag{15}$$

Otherwise, if $v \in \mathcal{V}(\mathcal{X}_i)$ is a vertex of $\mathcal{X}$, then it satisfies (13). Therefore, due to the convexity of $\ell(x)$ proved in Lemma 3.6 and (13), (14), (15), the following is obtained:

$$\begin{aligned} \lambda \ell(x_k) &= \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)(\lambda \ell(v)) \\ &\geq \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) \ell(Av + Bu^*(v) + w_k) \\ &\geq \ell(A \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) v + B \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) u^*(v) + w_k) \\ &= \ell(Ax_k + B \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) u^*(v) + w_k). \end{aligned} \tag{16}$$

Recall that $u^*(v) \in \mathbb{U}$, $\forall v \in \mathcal{V}(\mathcal{X}_i) \cap \mathcal{V}(\mathcal{X})$ and $u^*(v) = Kv \in \mathbb{U}$, $\forall v \in \mathcal{V}(\mathcal{X}_i) \cap \mathcal{V}(\Omega)$, then it follows that

$$\sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) u^*(v) \in \mathbb{U}. \tag{17}$$

Therefore, (17) leads to:

$$\ell(Ax_k + B \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) u^*(v) + w_k) \geq \ell(Ax_k + Bu^*(x_k) + w_k). \tag{18}$$

From (16) and (18), the following inclusion can be obtained:

$$\lambda \ell(x_k) \geq \ell(Ax_k + Bu^*(x_k) + w_k), \forall w_k \in \mathbb{W}. \tag{19}$$

Recall that $0 \leq \lambda < 1$, therefore

$$\ell(x_k) > \ell(Ax_k + Bu^*(x_k) + w_k), \forall w_k \in \mathbb{W}, \tag{20}$$

meaning $\{\ell(x_k)\}_{k=0}^{\infty}$ is a strictly decreasing series and bounded in the interval $[0, a]$. Thus, this series is convergent to 0. In other words, $\ell(x)$ serves as a Lyapunov function, defined in Definition 3.10. □

Note that in order to reduce the impact of the disturbances, Ω can be chosen as the minimal positively invariant set, approximated in e.g. Rakovic et al. [2005].

Remark 3.13. Note that by the construction, the partition associated with a convex lifting in Algorithm 1, may not be a decomposition of simplexes (e.g. triangles in $\mathbb{R}^2$). The design method in this paper does not rely on such a decomposition, but relies on a continuous, convex function, defined over such a decomposition. This approach is simple and needs only to solve a linear programming problem at each sampling instant. However, the associated control law is not continuous at the moment the state switches into Ω (see step 2 of Algorithm 2). As a future research problem, it would be interesting to relate convex lifting with the continuity of control function.

Remark 3.14. Another open problem is to guarantee closed-loop stability of the proposed method for a domain of attraction as the $N-$*step robust controllable set* denoted e.g. by $\mathcal{K}_N(\Omega)$, presented via Definition 2.9 in Kerrigan [2001]. Note that in this case, proving the strict decrease of $\ell(x)$ becomes more difficult. Also, this strict decrease may not be successive.

Remark 3.15. Unlike the proposed method, another design methodology, based on a triangulation of $\mathcal{K}_N(\Omega)\backslash\Omega$ into a Delaunay decomposition, can be found in Scibilia et al. [2009] (persistent disturbances were not taken into account in this study). This method approximates optimal solution to an MPC problem via a piecewise affine function defined over such a Delaunay decomposition. Moreover, another approach based on the interpolation of the exact control action at the vertices of the domain of attraction and a local control law over the unconstrained region, is presented in Nguyen et al. [2013]. This method introduces a discontinuous Lyapunov-like function. Whereas, this paper presents a continuous Lyapunov function.

4. NUMERICAL EXAMPLE

For illustration, the double integrator system is considered:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix} u_k + w_k \\ y_k &= [1 \ 0]\, x_k, \end{aligned} \tag{21}$$

subject to constraints:

$$-2 \le u_k \le 2, \ \begin{bmatrix} -20 \\ -20 \end{bmatrix} \le x_k \le \begin{bmatrix} 20 \\ 20 \end{bmatrix}, \ \|w_k\|_\infty \le 0.4.$$

A control law gain $K = [-0.8246 \ -1.5262]$ is chosen to compute the maximal disturbance invariant set Ω based on the algorithm proposed in Gilbert and Tan [1991]. Also, the maximal $0.9-$contractive set $P_{0.9}$ is computed, based on the algorithm in Blanchini [1994]. These two sets are shown in Fig. 1. A convex lifting $\ell(x)$ is shown in Fig. 2 according to Algorithm 1 with $a = 2$. Optimal controller which solves the linear programming problem (7), is presented in Fig. 3. Accordingly, the discontinuous change of $u^*(x_k)$ at instant 12 is due to the discontinuity of optimal control to (7) while switching into Ω. The closed loop dynamics shown in Fig. 4 illustrate the fact that this control law ensures the robust stability in the sense of Lyapunov. Finally, Fig. 5 visualizes the strict decrease of convex lifting $\ell(x)$ along the state over $\mathcal{X}\backslash\Omega$.

Algorithm 1 is carried out in the environment of MPT 3.0 Herceg et al. [2013].

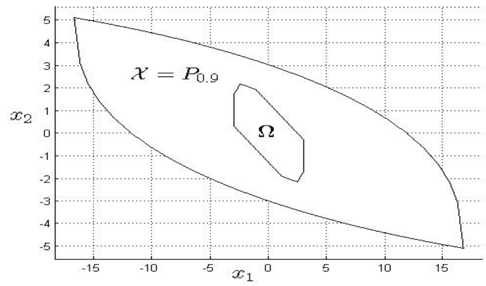

Fig. 1. The maximal disturbance invariant set Ω and domain of attraction $\mathcal{X} = P_{0.9}$.

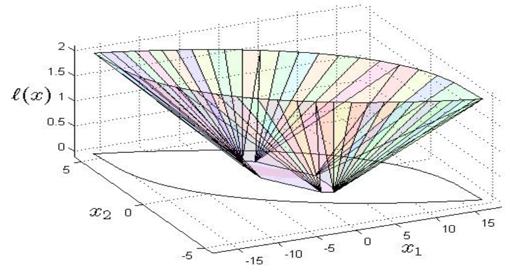

Fig. 2. A convex lifting $\ell(x)$ constructed by Algorithm 1 with $a = 2$.

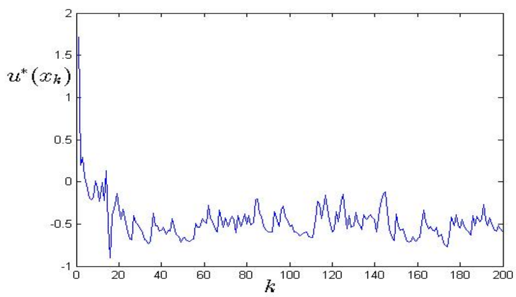

Fig. 3. Optimal control law solves the linear programming problem (7) with random behavior of disturbances.

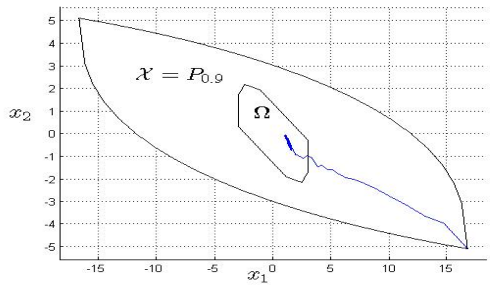

Fig. 4. Closed loop stability.

5. CONCLUSION

This paper presents a new method to design robust control law for constrained linear system affected by bounded additive disturbances. This method is based on convex liftings. It is shown to guarantee the recursive feasibility and also closed loop stability in the sense of Lyapunov.

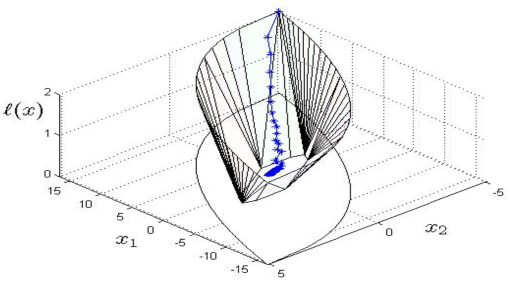

Fig. 5. The strict decrease of $\ell(x)$ over $\mathcal{X}\backslash\Omega$ along the state.

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