Fault Detection using Set-Membership Estimation based on Multiple Model Systems

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Abstract— This paper proposes a new Fault Detection algorithm based on Multiple Models approach for linear systems with bounded perturbations. The consistency of each model with the measurements is checked at each sample time based on set-membership state estimation. A Min-Max Model Predictive Control is developed in order to find the optimal control and the best model to use for the system in spite of the presence of component/actuator/sensor faults. An illustrative example is analyzed in order to show the effectiveness of the proposed approach.

Index Terms— Fault Detection, Multiple Models, set-membership state estimation, Min-Max MPC, bounded noises and perturbations, linear systems, quadratic programming.

I. INTRODUCTION

Fault Tolerant Control (FTC) is a new research area that can maintain an acceptable level of control even after the occurrence of faults. A generally accepted definition of a fault is that it is an intolerable deviation of at least one characteristic property or parameter of a system from its acceptable/usual/standard conditions. The determination of a fault at a certain time is referred to as Fault Detection (FD). This aims at developing Fault Detection methods. A pertinent overview and discussion of these methods can be found in [PFC00] and [BKLS03].

One of the many different approaches of FD is the Multiple Models (MM) technique. A Multiple Model technique consists in the construction of a set of models that contains local models corresponding to specific fault conditions of the monitored system [May99], [RL08]. The motivation for using Multiple Model systems for FD stems from the fact that a large class of fault conditions can be modeled, contrary to other FD methods that can only applied to limited types/number of faults conditions.

For linear systems (represented by an evolution matrix A, a control matrix B and an observation matrix C), the MM systems is an attractive technique for FD due to its flexible structure that allows us intuitive modeling of faults. In general, in a state-space representation, a component fault can be modeled by a modification of the A matrix, an actuator fault can be modeled by a change of the B

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matrix, and a sensor fault can be modeled by an alteration of the C matrix. For this reason FD using the MM systems has attracted significant interest [ZL98], [YH03], [VST04], [DG08].

Fault Detection using Multiple Models in context of Takagi-Sugeno approach has been explored in several works [HRMB12], [MKR07]. The authors of [HVBK06] propose a method for estimating both the weights and the state of a Multiple Model systems with one common state vector. In this system, the weights are related to the activation of each individual model. However, perturbations and measurement noises are assumed to be stochastic with a given covariance representation. The fault diagnosis method presented in [MTS05] is based on a generation of residual bank of robust parity spaces decoupled from the faults; but the residual is obtained using statistical method which sometimes makes difficult the parameters tuning.

In the works presented above, the perturbations are assumed having a known distribution. This assumption is in many cases difficult to validate. Thus, it may be more realistic to assume that the perturbations and measurement noises are unknown but bounded. This leads to use set-membership approaches for the estimation [Sch68], [BR71], [Che94], [FH82].

In this context, the current paper proposes a new Fault Detection using set-membership estimation approach based on Multiple Models technique. These models are constructed by referring to the original system, such that each model is adequate to one faulty mode. This method consists first in checking the consistency between each model with the available measurements. This consistency checking is based on a guaranteed ellipsoidal set-membership state estimation [BSA+14]. Second, the set of compatible models with the measurements is formed. In a third step, a Min-Max Model Predictive Control (MPC) [ARdlP05] is developed for each compatible model ensuring the desirable performances. A quadratic criterion is minimized in order to choose:

- The best control to be applied to the original system;
- The best model for the estimation.

The novelty in this paper is the use of set-membership estimation coupled with Min-Max MPC to estimate the state of linear systems with unknown but bounded perturbations and measurement noises despite the presence of component, actuator and sensor faults.

The remainder of this paper is organized as follows. Section II formulates the problem of Fault Detection in the context of linear systems with bounded perturbations and measurement noises. The ellipsoidal state estimation method

and the Fault Detection approach in the fault-free case are presented in Section III. Section IV focuses on the FD and FTC based on a Min-Max MPC problem. Section V proposes a detailed formulation of the Min-Max MPC problem. An illustrative example showing the performances of the proposed Fault Detection technique is proposed in Section VI. Finally, some concluding remarks and perspectives are drawn in Section VII.

Notations

An *interval* $[a, b]$ is defined as the set $\{x \in \mathbb{R} : a \leq x \leq b\}$. A *unitary interval* is $\mathbf{B} = [-1, 1]$. A *box* $([a_1, b_1], \ldots, [a_n, b_n])^\top$ is an interval vector. A *unitary box* in $\mathbb{R}^m$, denoted by $\mathbf{B}^m$, is a box composed by m unitary intervals.

A symmetric matrix $M = M^\top \in \mathbb{R}^{n \times n}$ is called a *positive definite matrix* (resp. *negative definite matrix*), denoted $M \succ 0$ (resp. $M \prec 0$), if $z^\top M z > 0$ (resp. $z^\top M z < 0$) for all non-zero vectors z with real entries ($z \in \mathbb{R}^n \backslash \{0_n\}$).

A bounded *ellipsoidal set* $\mathcal{E}(P, \bar{x}, \rho)$ is defined as:

$$\mathcal{E}(P, \bar{x}, \rho) = \{ x \in \mathbb{R}^n : (x - \bar{x})^\top P (x - \bar{x}) \leq \rho \}, \quad (1)$$

where $P = P^\top \succ 0$ is the shape matrix of the ellipsoid, $\bar{x} \in \mathbb{R}^{n_x}$ is its center and $\rho \in \mathbb{R}_+^*$ is its radius.

A *polyhedron* $\mathcal{P} \in \mathbb{R}^n$ is defined by a system of finitely many inequalities $Ax \leq b$ such that:

$$\mathcal{P} = \{ x \in \mathbb{R}^n : Ax \leq b \}. \quad (2)$$

Given a bounded polyhedral set $\mathcal{X}$, denote by $\mathcal{V}_\mathcal{X}$ the *set of its vertices*.

A *polytope* $\mathcal{P} \in \mathbb{R}^n$ is defined by a finite set $\mathcal{X} \subseteq \mathbb{R}^n$ such that:

$$\mathcal{P} = conv\ (\mathcal{X}). \quad (3)$$

A *strip* is defined as $\mathcal{S}(y, c, \sigma) = \{x \in \mathbb{R}^n : |c^\top x - y| \leq \sigma\}$.

Denote by $\mathcal{C}_M$ the set of compatible models with the measurements.

A matrix $\mathbb{O}_{n,m}$ defines a zeros matrix of dimensions $n \times m$.

A matrix $\mathbb{I}_n$ defines the identity matrix of dimensions $n \times n$.

A matrix $\mathbb{1}_{n,m}$ defines the matrix of dimensions $n \times m$ having all elements equal to 1.

II. PROBLEM FORMULATION

Consider the following discrete-time LTI (Linear Time Invariant) system:

$$\begin{cases} x_{k+1} = AG_{i_c} x_k + BH_{i_a} u_k + E\omega_k \\ y_k = CI_{i_s} x_k + F\omega_k \end{cases} \quad (4)$$

with $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$, $E \in \mathbb{R}^{n_x \times (n_x + n_y)}$, $F \in \mathbb{R}^{n_y \times (n_x + n_y)}$, $G_{i_c} \in \mathbb{R}^{n_x \times n_x}$, $H_{i_a} \in \mathbb{R}^{n_u \times n_u}$ and $I_{i_s} \in \mathbb{R}^{n_x \times n_x}$. $x_k \in \mathbb{R}^{n_x}$ is the state vector of the system, $u_k \in \mathbb{R}^{n_u}$ is the input vector, and $y_k \in \mathbb{R}^{n_y}$ is the measured output vector at sample time k. The vector $\omega_k \in \mathbb{R}^{n_x + n_y}$ contains the state perturbations and the measurement perturbations (noise, offset, etc.). The perturbations

are assumed to be bounded by unitary boxes $\omega_k \in \mathbf{B}^{n_x + n_y}$ for every $k \geq 0$. Consider that the initial state x_0 belongs to the ellipsoid $\mathcal{E}(P_0, \bar{x}_0, \rho_0) = \{ x \in \mathbb{R}^{n_x} : (x - \bar{x}_0)^\top P_0 (x - \bar{x}_0) \leq \rho_0 \}$.

The matrix G_{i_c}, with $i_c \in \mathbb{I}_c = \{0, 1, 2, \ldots, n_c\}$ and n_c denoting the number of the considered component faults, is a diagonal matrix modeling the i_c-th component mode. In a similar way, the matrix H_{i_a}, with $i_a \in \mathbb{I}_a = \{0, 1, 2, \ldots, n_a\}$ and n_a the number of considered actuator faults, is a diagonal matrix modeling the i_a-th actuator mode. The matrix I_{i_s}, with $i_s \in \mathbb{I}_s = \{0, 1, 2, \ldots, n_s\}$, where n_s denotes the number of considered sensor faults, is a diagonal matrix modeling the i_s-th sensor mode.

All diagonal entries of G_{i_c}, H_{i_a} and I_{i_s} belong to $[0, 1]$ where 0 or 1 means that the corresponding components, actuators and sensors are completely faulty or healthy, respectively. A value in the range $(0, 1)$ denotes a partial degradation of the corresponding components, actuators and sensors.

It is assumed that the pairs (AG_{i_c}, BH_{i_a}) and (AG_{i_c}, CI_{i_s}) are respectively stabilizable and detectable under all the considered modes.

Remark 1: The system (4) can be rewritten in the following form:

$$\begin{cases} x_{k+1} = A(x_k + f_{x_k}) + B(u_k + f_{u_k}) + E\omega_k \\ y_k = Cx_k + F\omega_k + f_{y_k} \end{cases} \quad (5)$$

where f_{x_k}, f_{u_k} and f_{y_k} are respectively the component fault, actuator fault and the sensor fault. It is easy to verify this, by taking $f_{x_k} = (G_{i_c} - \mathbb{I}_{n_x}) x_k$, $f_{u_k} = (H_{i_a} - \mathbb{I}_{n_u}) u_k$ and $f_{y_k} = (I_{i_s} - \mathbb{I}_{n_x}) x_k$.

Given an ellipsoidal estimation for x_k of the form $\mathcal{E}(P, \bar{x}_k, \rho_k)$, with P unknown and $k > 0$, the objective of this paper is to provide an ellipsoidal estimation for x_{k+1} of the form $\mathcal{E}(P, \bar{x}_{k+1}, \rho_{k+1})$ using the ellipsoidal set-membership state estimation presented in [BSA$^+$14] despite the presence of possible faults (on components, actuators or sensors).

The next section summarizes the ellipsoidal state estimation technique [BSA$^+$14] used for the fault-free case.

III. ELLIPSOIDAL STATE ESTIMATION OF THE FAULT-FREE SYSTEM

This subsection briefly describes the guaranteed ellipsoidal state estimation [BSA$^+$14] for the system (4) in the fault-free case (i.e. G_{i_c}, H_{i_a} and I_{i_s} are identity matrices). In this case, the system (4) becomes:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + E\omega_k, \\ y_k = Cx_k + F\omega_k. \end{cases} \quad (6)$$

The ellipsoidal estimation method is based on the minimization of the ellipsoidal radius at each iteration by solving a Linear Matrix Inequality (LMI) problem.

Consider an initial state vector $x_0 \in \mathcal{E}(P_0, \bar{x}_0, \rho_0)$ and assume that $x_k \in \mathcal{E}(P, \bar{x}_k, \rho_k)$ at time k. If there exist a matrix $Y_k \in \mathbb{R}^{n_x \times n_y}$, a matrix $S = S^\top \succ 0$ in $\mathbb{R}^{(n_x + n_y) \times (n_x + n_y + n)}$ and the scalars $\rho_{k+1} > 0$ and $\beta \in (0, 1)$ for which the

following LMI holds:

$$\min_{\beta, Y_k, S, \rho_{k+1}} \rho_{k+1}$$

subject to

$$\begin{cases} \begin{bmatrix} \beta P & * & * \\ PA - Y_k C & P & * \\ 0 & E^\top P - F^\top Y_k^\top & S \end{bmatrix} \succ 0, \\ \rho_{k+1} - \beta \rho_k > 0, \\ \beta < 1 \end{cases} \quad (7)$$

then the system state x_{k+1} at time $k+1$ is guaranteed to belong to the ellipsoid $\mathcal{E}(P, \bar{x}_{k+1}, \rho_{k+1}), \forall \omega_k \in \mathbf{B}^{n_x + n_y + n_\delta}$, with the following notations:

$$Y_k = PL_k, \quad (8)$$
$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + L_k(y_k - C\bar{x}_k). \quad (9)$$

The proof of this result is given in [BSA+14]. This method allows us to estimate the state of the system (4) in the fault-free case, offering a good trade-off between accuracy and computation time of the estimation.

Remark 2: At time $k = 1$, an initialization step is effectuated. Thus, given a scalar $\beta \in (0, 1)$, the shape matrix P is found by solving the following LMI problem (obtained from (7) with P as decision variable):

$$\min_{P, Y_k, S, \rho_{k+1}} \rho_{k+1}$$

subject to

$$\begin{cases} \begin{bmatrix} \beta P & * & * \\ PA - Y_k C & P & * \\ 0 & E^\top P - F^\top Y_k^\top & S \end{bmatrix} \succ 0, \\ \rho_{k+1} - \beta \rho_k > 0, \end{cases} \quad (10)$$

Once the matrix P is fixed, the LMI problem (7) is solved at each time instant.

The objective is twofold:

- Find the models which are compatible with the set of the measurements;
- Use this ellipsoidal estimation method to estimate the state of the system (4) despite the presence of faults.

This idea will be developed in the next subsection.

IV. MULTIPLE MODELS FAULT DETECTION

The idea is to construct a set of p Multiple Models $\mathcal{M} = \{M_1, M_2, \ldots, M_p\}$ such that M_1 represents the fault-free case, i.e. $A_1 = A$, $B_1 = B$, $C_1 = C$, $E_1 = E$ and $F_1 = F$. For for $i = 2, \ldots, p$, each model M_i is dedicated to one faulty mode. Note that the model M_i is defined by the matrices A_i, B_i, C_i, E_i and F_i.

The state of the system (4) is estimated by each model M_i based on the ellipsoidal estimation (7) presented in the previous section. Considering the presence of faults, the consistency between the model M_i and the measurement has to be checked at each sample time. Then, the objective is to find the models which are compatible with the set of measurements. Once this set is computed, a Min-Max Model Predictive Control is developed in order to stabilize the state x_k of the system (4) and to decide which is the best model to

estimate the state of the system for the next step. The details of the Min-Max MPC problem are given in Section V.

Algorithm 1 provides a general form of the Fault Detection and Fault Tolerant Control strategy based on checking consistency between the models and the measurements. The idea of this algorithm is summarized below:

- **Initialization:** (*step 1 to step 5*)
  The state estimation for is initialized by the ellipsoidal set $\mathcal{E}(P_0, \bar{x}_0, \rho_0)$ is *step 2*. The estimation for each model $M_i \in \mathcal{M}$, $i = 1, \ldots, p$, is also initialized by the same ellipsoidal set ($\mathcal{E}_i(P_0, \bar{x}_0, \rho_0) = \mathcal{E}(P_0, \bar{x}_0, \rho_0)$).

- **Compatible models set construction:** (*step 7 to step 18*)
  At each sample time k, the output measurement y_k in (4) obtained from the sensors is used to build the parametrized polytope $\mathcal{P}_{check}(C_i, y_k, F_i)$. This polytope corresponds to the consistent state set with the measurements y_k and the construction of this polytope is detailed in Appendix. The consistency between the ellipsoidal estimated set $\mathcal{E}_i(P, \bar{x}_{k,i}, \rho_{k,i})$ and the polytope $\mathcal{P}_{check}(C_i, y_k, F_i)$ is verified for each model $M_i \in \mathcal{M}$. The ellipsoidal set $\mathcal{E}_i(P, \bar{x}_{k,i}, \rho_{k,i})$ represents the state estimation with the model M_i.
  This consistency test is solved by the following Quadratic Programming (QP) optimization problem with linear constraints:

$$t^* = \min_{x_k \in \mathcal{E}_i(P, \bar{x}_k, \rho_k)} (x_k - \bar{x}_k)^\top P(x_k - \bar{x}_k)$$

  subject to

$$S x_k \leq T. \quad (11)$$

  If $t^* < \rho_k$, then the intersection $\mathcal{E}_i(P, \bar{x}_k, \rho_k) \cap \mathcal{P}_{check}(C_i, y_k, F_i)$ is not empty. Else, the intersection is empty $\mathcal{E}_i(P, \bar{x}_k, \rho_k) \cap \mathcal{P}_{check}(C_i, y_k, F_i) = \emptyset$.
  If the consistency is proved (i.e. non-empty intersection), the model M_i is called *compatible with the measurements* and it is added to the set $\mathcal{C}_M$ containing all the compatible models with the measurements. Otherwise, the model M_i is called *incompatible with the measurements*.

- **Construction of a Min-Max Model Predictive Control for each compatible model:** (*step 19 to step 24*)
  A sequence of control $\mathbf{u}_{k|k,j} = [u_{k|k,j}, u_{k+1|k,j}, \ldots, u_{k+h-1|k,j}]^\top$ is computed for each model $M_j \in \mathcal{C}_M$, with $j = 1, \ldots, s_M$ (s_M is the size of $\mathcal{C}_M$), by minimizing the following criterion:

$$\mathbf{u}_{k|k,j} = \arg \min_{\mathbf{u}_{k|k,j}} \max_{\omega_k \in \mathbf{B}^{n_x + n_y}} J_j(u_{k|k,j}, \omega_{k|k,j}, x_{k|k,j}), \quad (12)$$

  subject to

$$x_{k+l|k} \in \mathbb{X} \text{ for } l = 1, \ldots, h$$
$$u_{k+l|k} \in \mathbb{U} \text{ for } l = 1, \ldots, h$$

  where h is the prediction horizon, $x_{k+l|k}$ represents the prediction of the state for the sample time $k + l$ at the sample time k, $u_{k+l|k}$ is the control prediction for the sample time $k + l$ at the sample time k, $\omega_{k+l|k}$ is the

perturbation prediction for the sample time $k + l$ at the sample time k and the cost function is defined as:

$$J_j(u_{k|k,j}, \omega_{k|k,j}, x_{k|k,j}) =$$
$$= \sum_{l=0}^{h-1} \left(x_{k+l|k,j}^\top Q x_{k+l|k,j}^\top + u_{k+l|k,j}^\top R u_{k+l|k,j} \right). \tag{13}$$

The cost function $J_j(u_{k|k,j}, \omega_{k|k,j}, x_{k|k,j})$ is maximized with respect to $\omega_{k+l|k,j} \in \mathbf{B^{n_x + n_y}}$ (corresponding to the worst case situation) and minimized with respect to $u_{k+l|k,j}$. The index j refers to the model $M_j \in \mathcal{C}_M$. Generally, the constraints on the state and input vectors and the choice of the weighting matrices Q and R are due to physical, safety or performance considerations. Then, the set of controllers $\mathcal{U}_k = \{\mathbf{u}_{k|k,1}, \ldots, \mathbf{u}_{k|k,s_M}\}$ suitable for each model $M_j \in \mathcal{C}_M$ is constructed.

- **Computing the optimal control and the best model for the estimation:** (*step 25*)

  The objective is to determine the best control $\mathbf{u}_{k|k,j}^* \in \mathcal{U}_k$ for the system (4) and the best model $M_j^* \in \mathcal{C}_M$ to use for the estimation. For this, the following optimization problem is solved:

$$(\mathbf{u}_{k|k}^*, M_{k|k}^*) = \arg \min_{\mathbf{u}_{k|k} \in \mathcal{U}_k} \max_{M_j \in \mathcal{C}_M} J(u_{k|k}, \omega_{k|k}, x_{k|k}), \tag{14}$$

  with the cost function $J(u_{k|k}, \omega_{k|k}, x_{k|k}) = \sum_{l=0}^{h-1} \left(x_{k+l|k}^\top Q x_{k+l|k}^\top + u_{k+l|k}^\top R u_{k+l|k} \right)$.

  Based on the receding horizon strategy, the control $u_{k|k}^*$ that will be applied to the system (4) is given by the first n_u components of the control sequence $\mathbf{u}_{k|k}^*$ as follows:

$$u_{k|k}^* = \begin{bmatrix} \mathbb{I}_{n_u} & \mathbb{O}_{n_u, ((N_{pred}-1)n_u)} \end{bmatrix} \mathbf{u}_{k|k}^*. \tag{15}$$

- **Computing the estimation for each model:** (*step 26 to step 32*)

  It consists in computing the ellipsoidal estimated sets $\mathcal{E}_i(P, \bar{x}_{k+1,i}, \rho_{k+1,i})$ for each model $M_i \in \mathcal{M}$, for $i = 1, \ldots, p$. If the model M_i was compatible with the measurement y_k (i.e. $M_i \in \mathcal{C}_M$), then the ellipsoidal estimation set $\mathcal{E}_i(P, \bar{x}_{k+1,i}, \rho_{k+1,i})$ is computed according to (7) using the model M_i, the control $u_{k|k}^*$ and the measurement y_k. Otherwise, the ellipsoidal estimation set $\mathcal{E}_i(P, \bar{x}_{k+1,i}, \rho_{k+1,i})$ is computed according to (7) using the best model M_k^*, the control $u_{k|k}^*$ and the measurement y_k.

- **Obtained estimation:** (*step 33*)

  Finally, the ellipsoidal estimation set $\mathcal{E}(P, \bar{x}_{k+1}, \rho_{k+1})$ is based on the best model M_k^*, the optimal control $u_{k|k}^*$ and y_k.

Algorithm 1. Fault Detection

1. $k \leftarrow 0$;
2. $\mathcal{E}(P_0, \bar{x}_0, \rho_0) \leftarrow \{x \in \mathbb{R}^{n_x} : (x - \bar{x}_0)^\top P_0(x - \bar{x}_0) \leq \rho_0\}$;
3. **for** $i = 1 : p$
4. $\mathcal{E}_i(P_0, \bar{x}_0, \rho_0) = \mathcal{E}(P_0, \bar{x}_0, \rho_0)$;
5. **end for**

6. **for** $k = 0 : N - 1$
7. $\mathcal{C}_M = \emptyset$;
8.   Collect y_k;
9.   **for** $i = 1 : p$
10.   Use the output measurements y_k to construct the polytope $\mathcal{P}_{check}(C_i, y_k, F_i)$;
11.   **if** $\mathcal{E}_i(P, \bar{x}_k, \rho_k) \cap \mathcal{P}_{check}(C_i, y_k, F_i) = \emptyset$
12.   The model M_i is not compatible with the set of measurements;
13. $\mathcal{C}_M = \mathcal{C}_M$;
14.   **else**
15.   The model M_i is compatible with the set of measurements;
16. $\mathcal{C}_M = \{\mathcal{C}_M, M_i\}$;
17.   **end if**
18.   **end for**
19. $s_M = \text{size}(\mathcal{C}_M)$
20. $\mathcal{U}_k = \emptyset$
21.   **for** $j = 1 : s_M$
22.   Compute $\mathbf{u}_{k|k,j}$ by solving the criterion (12);
23. $\mathcal{U}_k = \{\mathcal{U}_k, \mathbf{u}_{k|k,j}\}$;
24.   **end for**
25.   Compute $u_{k|k}^*$ and M_k^* using (14) and (15);
26.   **for** $i = 1 : p$
27.   **if** $M_i \in \mathcal{C}_{\mathcal{M}}$
28.   Compute the ellipsoidal estimation set $\mathcal{E}_i(P, \bar{x}_{k+1}, \rho_{k+1})$ according to (7) using the model M_i (M_i is defined by the matrices A_i, B_i, C_i, E_i and F_i) and the control $u_{k|k}^*$;
29.   **else**
30.   Compute the ellipsoidal estimation set $\mathcal{E}_i(P, \bar{x}_{k+1}, \rho_{k+1})$ according to (7) using M_k^* and $u_{k|k}^*$;
31.   **end if**
32.   **end for**
33.   Compute the ellipsoidal estimation set $\mathcal{E}(P, \bar{x}_{k+1}, \rho_{k+1})$ according to (7) using the model M_k^*;
34. **end for**.

V. MIN-MAX MODEL PREDICTIVE CONTROL

This section details the development of Min-Max Model Predictive Control applied to each model M_j belonging to the compatible set $\mathcal{C}_M$. The control signal is found by minimizing a worst case (with respect to the perturbations ω_k) of a quadratic criterion (12). The Min-Max optimization problem (12) is reformulated as a quadratic programming (QP) problem. The controller is, then, computed using the ellipsoidal state estimation at the previous sample time by solving a simple QP problem.

Starting from the quadratic cost function:

$$J_j(u_{k,j}, \omega_k, x_{k,j}) = \sum_{l=0}^{h-1} \left(x_{k+l,j}^\top Q x_{k+l,j}^\top + u_{k+l,j}^\top R u_{k+l,j} \right) \tag{16}$$

the following state equations[1] are computed for each compatible model $M_j \in \mathcal{C}_M$:

$$\begin{cases} x_{k+1,j} = A_j x_{k,j} + B_j u_{k,j} + E_j \omega_{k,j}, \\ \vdots \\ x_{k+l,j} = A_j^l x_{k,j} + A_j^{l-1} B_j u_{k,j} + A_j^{l-2} B_j u_{k+1,j} + \\ \qquad\qquad + \ldots + B_j u_{k+l-1,j} + A_j^{l-1} F_j \omega_{k,j} + \\ \qquad\qquad + A_j^{l-2} F_j \omega_{k+1,j} + \ldots + F_j \omega_{k+l-1,j}, \\ \vdots \\ x_{k+h,j} = A_j^h x_{k,j} + A_j^{h-1} B_j u_{k,j} + A_j^{h-2} B_j u_{k+1,j} + \\ \qquad\qquad + \ldots + B_j u_{k+h-1,j} + A_j^{h-1} F_j \omega_{k,j} + \\ \qquad\qquad + A_j^{h-2} F_j \omega_{k+1,j} + \ldots + F_j \omega_{k+h-1,j}, \end{cases}$$

with h the prediction horizon and $j = 1, \ldots, s_M$. Denote by $\mathbf{u}_{k|k,j} = [u_{k|k,j}, u_{k+1|k,j}, \ldots, u_{k+h-1|k,j}]^\top$ and $\boldsymbol{\omega}_{k|k,j} = [\omega_{k|k,j}, \omega_{k+1|k,j}, \ldots, \omega_{k+h-1|k,j}]^\top$ the sequences of control signals and perturbations, respectively. Then, the state equation predicted for time $k+l$ at time k of the model $M_j \in \mathcal{C}_M$ can be rewritten as:

$$x_{k+l|k,j} = A_j^l x_{k|k,j} + \mathcal{A}_{l,j} B_j \mathbf{u}_{k|k,j} + \mathcal{A}_{l,j} \boldsymbol{\omega}_{k|k,j} \quad (17)$$

where the $\mathcal{A}_{l,j}$ matrix is defined as:

$$\mathcal{A}_{l,j} = \begin{bmatrix} A_j^{l-1} & A_j^{l-2} & \ldots & A_j^0 & Z_l \end{bmatrix}$$

with $Z_l = \underbrace{\begin{bmatrix} \mathbb{0}_{n_x, n_x} & \ldots & \mathbb{0}_{n_x, n_x} \end{bmatrix}}_{h-j \text{ times}}$.

Replacing (17) in (16) and after some manipulations, the optimization problem (12) becomes:

$$\mathbf{u}_{k,j} = \arg \min_{\mathbf{u}_{k|k,j} \in \mathcal{U}_k} \max_{\boldsymbol{\omega}_{k|k,j} \in \mathbf{B}^{h \times (n_x + n_y)}} f(\mathbf{u}_{k|k,j}, \boldsymbol{\omega}_{k|k,j}), \quad (18)$$

where $f(\mathbf{u}_{k|k,j}, \boldsymbol{\omega}_{k|k,j}) = \alpha_1 + \alpha_2 \boldsymbol{\omega}_{k|k,j} + \alpha_3 \mathbf{u}_{k|k,j} + \boldsymbol{\omega}_{k|k,j}^\top \alpha_4 \boldsymbol{\omega}_{k|k,j} + \boldsymbol{\omega}_{k|k,j}^\top \alpha_5 \mathbf{u}_{k|k,j} + \mathbf{u}_{k|k,j}^\top \alpha_6 \mathbf{u}_{k|k,j}$ with

$$\begin{cases} \alpha_1 = x_{k|k,j}^\top \sum_{l=0}^{h-1} A_j^{l\top} Q A_j^l x_{k|k,j} \\ \alpha_2 = 2 x_{k|k,j}^\top \sum_{l=0}^{h-1} A_j^{l\top} Q \mathcal{A}_{l,j} \bar{F} \\ \alpha_3 = 2 x_{k|k,j}^\top \sum_{l=0}^{h-1} A_j^{l\top} Q \mathcal{A}_{l,j} \bar{B} \\ \alpha_4 = \bar{F}^\top \sum_{l=0}^{h-1} \mathcal{A}_{l,j}^\top Q \mathcal{A}_{l,j} \bar{F} \\ \alpha_5 = 2 \bar{F}^\top \sum_{l=0}^{h-1} \mathcal{A}_{l,j}^\top Q \mathcal{A}_{l,j} \bar{B} \\ \alpha_6 = \bar{F}^\top \sum_{l=0}^{h-1} \mathcal{A}_{l,j}^\top Q \mathcal{A}_{l,j} \bar{F} + \bar{R} \end{cases} \quad (19)$$

with $\bar{B} = \text{diag}(\underbrace{B, \ldots, B}_{h \text{ times}})$, $\bar{F} = \text{diag}(\underbrace{F, \ldots, F}_{h \text{ times}})$ and $\bar{R} = \text{diag}(\underbrace{R, \ldots, R}_{h \text{ times}})$.

The function $f(\mathbf{u}_{k|k,j}, \boldsymbol{\omega}_{k|k,j})$ is quadratic with respect to $\mathbf{u}_{k|k,j}$ and $\boldsymbol{\omega}_{k|k,j}$. In [ARdlP05], it is shown that the Min-Max MPC problem (18) is equivalent to:

$$\mathbf{u}_{k,j} = \arg \min_{\mathbf{u}_{k|k,j} \in \mathcal{U}_k} \max_{\boldsymbol{\omega}_{k|k,j} \in \mathcal{V}_{\mathbf{B}^{h \times (n_x + n_y)}}} f(\mathbf{u}_{k|k,j}, \boldsymbol{\omega}_{k|k,j}). \quad (20)$$

The problem (20) becomes a QP problem as follows:

$$\mathbf{u}_{k,j} = \arg \min_{\mathbf{u}_{k|k,j} \in \mathcal{U}_k} \tilde{f}(\mathbf{u}_{k|k,j}), \quad (21)$$

such that $\tilde{f}(\mathbf{u}_{k|k,j})$ is quadratic with respect to $\mathbf{u}_{k|k,j}$. In general the constraints $x_k \in \mathbb{X}$ and $u_k \in \mathbb{U}$ are given in the following form: $x_{\min} \leq x_k \leq x_{\min}$ and $u_{\min} \leq u_k \leq u_{\min}$. Finally, the problem (12) to solve is a QP problem:

$$\min_{\mathbf{u}_{k|k,j}} \tilde{f}(\mathbf{u}_{k|k,j})$$

subject to

$$\begin{bmatrix} \mathcal{A}_{l,j} \bar{B} \\ -\mathcal{A}_{l,j} \bar{B} \\ \mathcal{I}_l \\ -\mathcal{I}_l \end{bmatrix} \mathbf{u}_{k|k,j} \prec \begin{bmatrix} b_1 \\ b_2 \\ u_{\max} \\ -u_{\min} \end{bmatrix}, \quad (22)$$

for $l = 1, \ldots, h$ with $b_1 = x_{\max} - A_j^l x_{k|k} - \mathcal{A}_{l,j} \bar{F} \omega_{k|k}$ and $b_2 = -x_{\min} + A_j^l x_{k|k} + \mathcal{A}_{l,j} \bar{F} \omega_{k|k}$, $\forall \omega_{k|k} \in \mathcal{V}_{\mathbf{B}^{h \times (n_x + n_y)}}$ and $\mathcal{I}_l = \begin{bmatrix} \mathbb{0}_{l-1, n_u} & \mathbb{1}_{1, n_u} & \mathbb{0}_{h-l, n_u} \end{bmatrix}$.

Remark 3: The state $x_{k|k}$ is chosen equal to the nominal state which is in this case the center of the ellipsoidal state estimation set $\bar{x}_k$.

In the next section, an illustrative example showing the effectiveness of the *Algorithm 1* is presented.

VI. ILLUSTRATIVE EXAMPLE

Consider the following LTI discrete-time system:

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} x_k + \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} u_k + \\ \qquad + \begin{bmatrix} 0.05 & 0 & 0 & 0 \\ 0 & 0.02 & 0 & 0 \end{bmatrix} \omega_k \\ y_k = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix} \omega_k \end{cases} \quad (23)$$

with $\|\omega_k\|_\infty \leq 1$. The value of ω_k is randomly generated. The initial state belongs to the ellipsoid $\mathcal{E}(\mathbb{I}_2, [0 \ 0]^\top, 1)$. In this example, 4 models are considered. M_1 corresponds to the fault-free system, i.e. $A_1 = A$, $B_1 = B$, $C_1 = C$, $E_1 = E$ and $F_1 = F$. M_2 models the system with a component fault: $A_2 = \begin{bmatrix} 0.4 & 0.8 \\ 0.1 & 0.2 \end{bmatrix}$, $B_2 = B$, $C_2 = C$, $E_2 = E$ and $F_2 = F$. M_3 corresponds to an actuator fault, with $A_3 = A$, $B_2 = \begin{bmatrix} 0.15 \\ 0.1 \end{bmatrix}$, $C_3 = C$, $E_3 = E$ and $F_3 = F$. M_4 corresponds to the system having a partial fault in the second sensor. $A_4 = A$, $B_4 = B$, $C_4 = \begin{bmatrix} -2 & 1 \\ 0.5 & 0.5 \end{bmatrix}$, $E_4 = E$ and $F_4 = F$. The length simulation $N = 100$. The prediction horizon $h = 10$, the weighting matrices $Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ and $R = 5$. The following constraints are considered on

the state $x_{\min} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $x_{\max} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and on the input $u_{\min} = -0.8$ and $u_{\max} = 0.8$.

The simulated faults are described in Table I.

TABLE I
SIMULATED FAULT SCENARIO

Fault description	Time interval (samples)
50% fault in actuator	$10 - 20$
50% fault in sensor	$50 - 60$

Figures 1, 2 illustrate the bounds of x_1 and x_2 after 100 iterations. The solid blue lines represent the bounds obtained by *Algorithm 1*. The red stars represent the real state of the system (situated inside the estimated bounds). The state estimation is guaranteed despite the presence of the considered faults, however the bounds of the estimation set are larger when faults occur (compared to a fault-free time intervals).
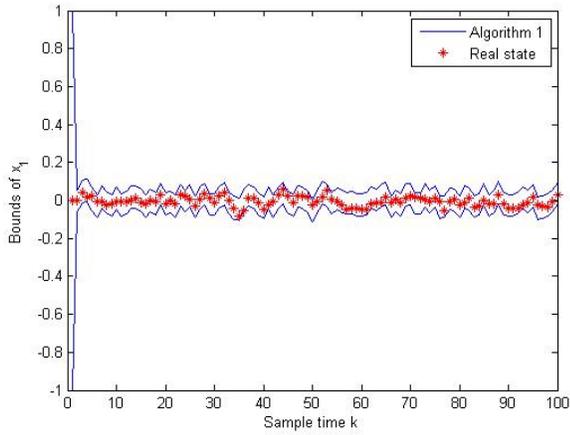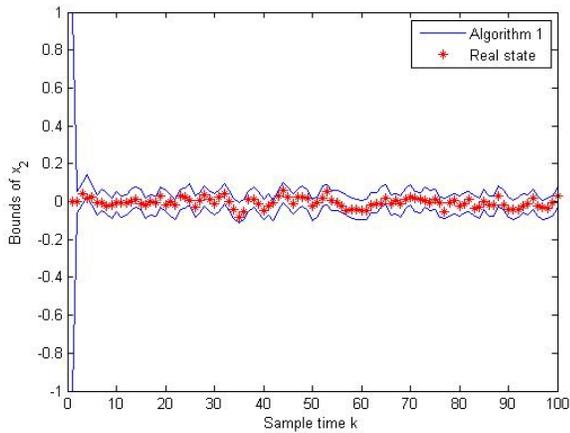

Fig. 1.   Bounds of x_1

Fig. 2.   Bounds of x_2

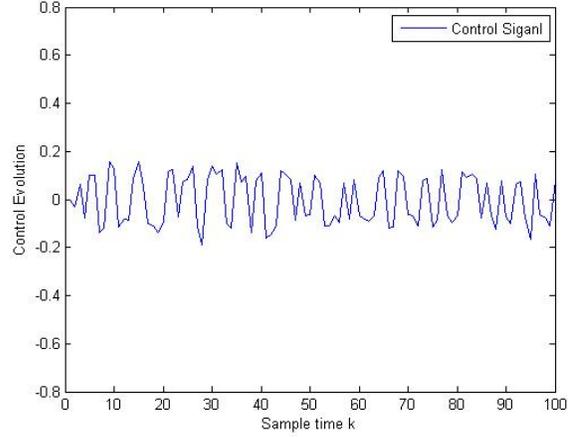Figure 3 represents the control u_k. The constraint $u_{\min} \leq u_k \leq u_{\max}$ is satisfied.

Fig. 3.   Evolution of the control u

Figures 4, 5, 6, 7 represent the fault signal obtained by models M_1, M_2, M_3 and M_4. When the fault signal is equal to 0 (respectively 1), the model M_i is compatible (respectively incompatible) with the measurements. Effectively, the model M_1 corresponding to the fault-free case system is compatible with the measurement when there is no fault. Even if for the considered actuator fault (between $10 - 20$ samples), the models M_2, M_3 and M_4 are compatible with the measurements, the optimal model chosen by the Min-Max MPC is M_3. In a similar way, the model M_4 is the optimal model for the considered sensor fault. This confirms the performance of *Algorithm 1*.
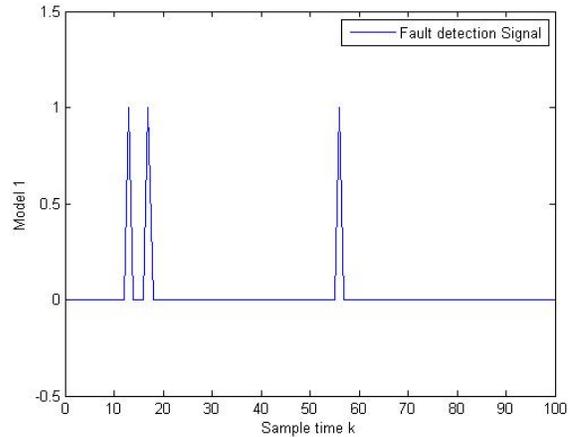

Fig. 4.   Fault signal for model M_1

VII. CONCLUSION

A new Fault Detection algorithm based on Multiple Models for linear systems with bounded perturbations and measurement noises has been proposed. The proposed algorithm allows to estimate the state of the system despite the presence of faults. A Min-Max MPC based on the ellipsoidal
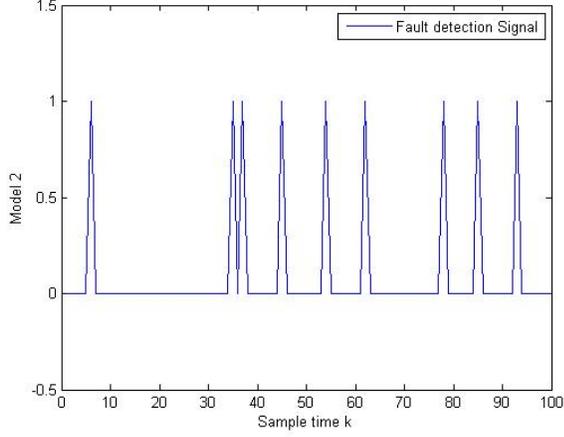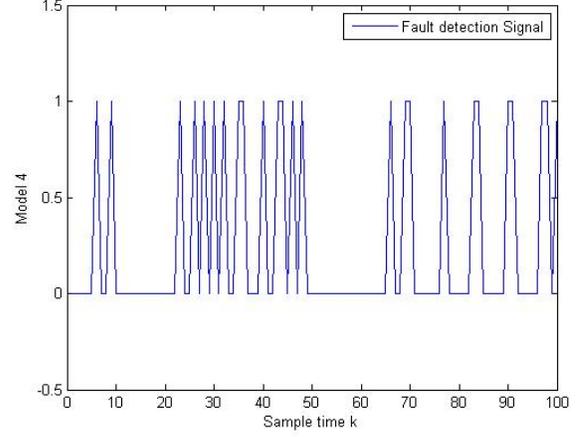
Fig. 5.   Fault signal for model M_2
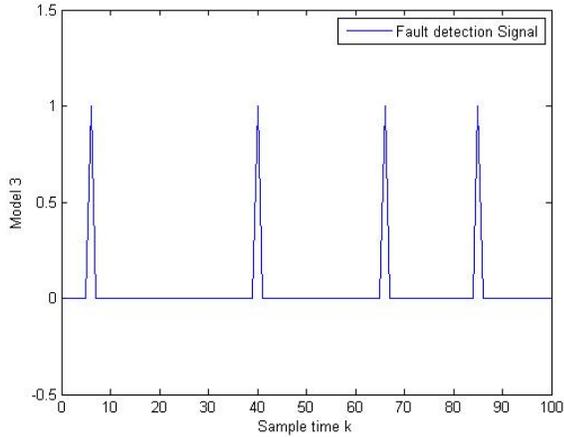

Fig. 7.   Fault signal for model M_4

Fig. 6.   Fault signal for model M_3

with the matrices $S = \begin{bmatrix} C_i \\ -C_i \end{bmatrix}$, $T = \begin{bmatrix} y_k + \mathbf{F}_i \\ -y_k + \mathbf{F}_i \end{bmatrix}$ and $\mathbf{F}_i = \begin{bmatrix} \|F_{i,1}\|_1 \\ \vdots \\ \|F_{i,n_y}\|_1 \end{bmatrix}$. Note that $F_{i,j}$ designs the jth line of the F_i matrix for the model M_i.

state estimation has been used. An example illustrates the effectiveness of the proposed method.

An interesting perspective is to extend this to the case of systems with interval uncertainties.

APPENDIX

This part details the construction of the polytope $\mathcal{P}_{check}(C_i, y_k, F_i)$, obtained from the intersection of all the n_y measurement strips. Each strip is defined by these two inequalities:

$$\begin{cases} C_{i,j} \leq y_{k,j} + \|F_{i,j}\|_1 \\ -C_{i,j} \leq -y_{k,j} + \|F_{i,j}\|_1 \end{cases} \tag{24}$$

such that i represents the ith model and j represents the jth line of C_i, F_i and y_k.

Then, the polytope $\mathcal{P}_{check}(C_i, y_k, F_i)$ is defined by the following constraints:

$$\mathcal{P}_{check}(C_i, y_k, F_i) = \{x \in \mathbb{R}^{n_x} : Sx_k \leq T\},$$

REFERENCES

[ARdlP05] T. Alamo, D. R. Ramirez, and D. .Munoz de la Pena. Min-Max MPC using a tractable QP problem. In *Proc. of the 49th IEEE Conference on Decision and Control, and the European Control Conference*, 2005. Seville, Spain.

[BKLS03] M. Blanke, M. Kinnaert, J. Lunze, and M. Staroswiecki. Diagnosis and Fault-Tolerant Control. *Springer-Verlag, Berlin Heidelberg*, 2003.

[BR71] D. P. Bertsekas and I. B. Rhodes. Recursive state estimation for a set-membership description of uncertainty. *IEEE Trans. Automat. Contr.*, 16(2):117–128, 1971.

[BSA+14] S. Ben Chabane, C. Stoica Maniu, T. Alamo, E.F. Camacho, and D. Dumur. A new approach for guaranteed ellipsoidal state estimation. In *Proc. of the 19th World Congress IFAC*, Cape Town, 2014.

[Che94] F. L. Chernousko. *State estimation for dynamic systems*. CRC Press, Boca Raton, 1994.

[DG08] G. Ducard and H. P. Geering. Efficient nonlinear actuator fault detection and isolation system for unmanned aerial vehicles. *Journal of Guidance, Control, and Dynamics*, 31(1):225–237, 2008.

[FH82] Eli Fogel and Y. F. Huang. On the value of information in system identification-bounded noise case. *Automatica*, 18, 1982.

[HRMB12] H. Hamdi, M. Rodrigues, C. Mechmeche, and N. Benhadj Braiek. Robust Fault Detection and estimation for descriptor systems based on Multi-Models concept. *International Journal of Control, Automation, and Systems, IJCAS*, 10(6):1260–1266, 2012.

[HVBK06] R. Hallouzi, M. Verhaegen, R. Babuska, and S. Kanev. Model weight and state estimation for Multiple Model systems applied to Fault Detection and Identification. In *Proc. of the 14th IFAC Symposium on System Identification*, Newcastle, Australia, 2006.

[May99] P. S. Maybeck. Multiple model adaptive algorithms for detecting and compensating sensor and actuator/surface failures in aircraft flight control systems. *International Journal of Robust and Nonlinear Control*, 9(14):1051–1070, 1999.

[MKR07]   B. Marx, D. Koenig, and J. Ragot. Design of observers for Takagi-Sugeno nonlinear descriptor systems with unknown inputs and application to fault diagnosis. *IET Control Theory Application*, 1:1487–1495, 2007.

[MTS05]   M. A. Medina, D. Theilliol, and D. Sauter. Simultaneous fault diagnosis and robust activation function estimation of linear multiple-models considering the bounded modelling uncertainty. In *Mediterranean Conference on Control and Automation*, pages 405–410, 2005. Limassol, Cyprus.

[PFC00]   R. J. Patton, P. M. Frank, and R. N. Clark. Issues of fault diagnosis for dynamic systems. *Springer-Verlag London Ltd*, 2000.

[RL08]   J. Ru and X. R. Li. Variable-structure multiple-model approach to fault detection, identification, and estimation. *IEEE Transaction on Control System Technology*, 2008.

[Sch68]   F. C. Schweppe. Recursive state estimation: Unknown but bounded errors and system inputs. *IEEE Trans. Automat. Contr.*, 13(1):22–28, 1968.

[VST04]   V. G. Verma, R. Simmons, and S. Thrun. Real-time fault diagnosis. *IEEE Robotics and Automation Magazine*, 11(2):56–66, 2004.

[YH03]   G. G. Yen and L. W. Ho. Online multiple-model-based fault diagnosis and accommodation. *IEEE Transactions on Industrial Electronics*, 50(2):296–312, 2003.

[ZL98]   Y. Zhang and X. R. Li. Detection and diagnosis of sensor and actuator failures using IMM estimator. *IEEE Transactions on Aerospace and Electronic Systems*, 34(4):1293–1313, 1998.