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To cite this version:
Sofiane Ben Chabane, Cristina Stoica Maniu, E.F. Camacho, T. Alamo, Didier Dumur. Fault Detection using Set-Membership Estimation based on Multiple Model Systems. 2015. hal-01180982

HAL Id: hal-01180982
https://hal-supelec.archives-ouvertes.fr/hal-01180982
Submitted on 28 Jul 2015

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Fault Detection using Set-Membership Estimation based on Multiple Model Systems

S. Ben Chabane, C. Stoica Maniu, E.F. Camacho, T. Alamo, D. Dumur

Abstract—This paper proposes a new Fault Detection algorithm based on Multiple Models approach for linear systems with bounded perturbations. The consistency of each model with the measurements is checked at each sample time based on set-membership state estimation. A Min-Max Model Predictive Control is developed in order to find the optimal control and to use the system in spite of the presence of component/actuator/sensor faults. An illustrative example is analyzed in order to show the effectiveness of the proposed approach.

Index Terms—Fault Detection, Multiple Models, set-membership state estimation, Min-Max MPC, bounded noises and perturbations, linear systems, quadratic programming.

I. INTRODUCTION

Fault Tolerant Control (FTC) is a new research area that can maintain an acceptable level of control even after the occurrence of faults. A generally accepted definition of a fault is that it is an intolerable deviation of at least one characteristic property or parameter of a system from its acceptable/usual/standard conditions. The determination of a fault at a certain time is referred to as Fault Detection (FD). This aims at developing Fault Detection methods. A pertinent overview and discussion of these methods can be found in [PFC00] and [BKLS03].

One of the many different approaches of FD is the Multiple Models (MM) technique. A Multiple Model technique consists in the construction of a set of models that contains local models corresponding to specific fault conditions of the monitored system [May99], [RL08]. The motivation for using Multiple Model systems for FD stems from the fact that a large class of fault conditions can be modeled, contrary to other FD methods that can only be applied to limited types/number of faults conditions.

For linear systems (represented by an evolution matrix \( A \), a control matrix \( B \) and an observation matrix \( C \)), the MM systems is an attractive technique for FD due to its flexible structure that allows us intuitive modeling of faults. In general, in a state-space representation, a component fault can be modeled by a modification of the \( A \) matrix, an actuator fault can be modeled by a change of the \( B \) matrix, and a sensor fault can be modeled by an alteration of the \( C \) matrix. For this reason FD using the MM systems has attracted significant interest [ZL98], [YH03], [VST04], [DG08].

Fault Detection using Multiple Models in context of Takagi-Sugeno approach has been explored in several works [HRMB12], [MKR07]. The authors of [HVBMK06] propose a method for estimating both the weights and the state of a Multiple Model systems with one common state vector. In this system, the weights are related to the activation of each individual model. However, perturbations and measurement noises are assumed to be stochastic with a given covariance representation. The fault diagnosis method presented in [MTS05] is based on a generation of residual bank of robust parity spaces decoupled from the faults; but the residual is obtained using statistical method which sometimes makes difficult the parameters tuning.

In the works presented above, the perturbations are assumed having a known distribution. This assumption is in many cases difficult to validate. Thus, it may be more realistic to assume that the perturbations and measurement noises are unknown but bounded. This leads to use set-membership approaches for the estimation [Sch68], [BR71], [Che94], [FH82].

In this context, the current paper proposes a new Fault Detection using set-membership estimation approach based on Multiple Models technique. These models are constructed by referring to the original system, such that each model is adequate to one faulty mode. This method consists first in checking the consistency between each model with the available measurements. This consistency checking is based on a guaranteed ellipsoidal set-membership state estimation [BSA14]. Second, the set of compatible models with the measurements is formed. In a third step, a Min-Max Model Predictive Control (MPC) [ARdlP05] is developed for each compatible model ensuring the desirable performances. A quadratic criterion is minimized in order to choose:

- The best control to be applied to the original system;
- The best model for the estimation.

The novelty in this paper is the use of set-membership estimation coupled with Min-Max MPC to estimate the state of linear systems with unknown but bounded perturbations and measurement noises despite the presence of component, actuator and sensor faults.

The remainder of this paper is organized as follows. Section II formulates the problem of Fault Detection in the context of linear systems with bounded perturbations and measurement noises. The ellipsoidal state estimation method
and the Fault Detection approach in the fault-free case are presented in Section III. Section IV focuses on the FD and FTC based on a Min-Max MPC problem. Section V proposes a detailed formulation of the Min-Max MPC problem. An illustrative example showing the performances of the proposed Fault Detection technique is proposed in Section VI. Finally, some concluding remarks and perspectives are drawn in Section VII.

Notations

An interval \([a, b]\) is defined as the set \(\{x \in \mathbb{R} : a \leq x \leq b\}\). A unitary interval is \(B = [-1, 1]\). A box \([a_1, b_1], \ldots, [a_n, b_n]\) is a box composed by \(m\) unitary intervals.

A symmetric matrix \(M = M^T \in \mathbb{R}^{n \times n}\) is called a positive definite matrix (resp. negative definite matrix), denoted \(M > 0\) (resp. \(M < 0\)), if \(z^TMz > 0\) (resp. \(z^TMz < 0\)) for all non-zero vectors \(z\) with real entries \((z \in \mathbb{R}^n \setminus \{0\})\).

A bounded ellipsoidal set \(E(P, x, \rho)\) is defined as:
\[
E(P, x, \rho) = \{x \in \mathbb{R}^n : (x - \bar{x})^T P (x - \bar{x}) \leq \rho \}
\]
where \(P = P^T > 0\) is the shape matrix of the ellipsoid, \(\bar{x} \in \mathbb{R}^n\) is its center and \(\rho \in \mathbb{R}_+\) is its radius.

A polyhedron \(\mathcal{P} \in \mathbb{R}^n\) is defined by a system of finitely many inequalities \(Ax \leq b\) such that:
\[
\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b\}
\]

Given a bounded polyhedral set \(\mathcal{X}\), denote by \(\mathcal{V}_\mathcal{X}\) the set of its vertices.

A polytope \(\mathcal{P} \in \mathbb{R}^n\) is defined by a finite set \(\mathcal{X} \subseteq \mathbb{R}^n\) such that:
\[
\mathcal{P} = \text{conv}(\mathcal{X})
\]

A strip is defined as \(S(y, c, \sigma) = \{x \in \mathbb{R}^n : |c^T x - y| \leq \sigma\}\).

Denote by \(\mathcal{C}_M\) the set of compatible models with the measurements.

A matrix \(\mathbb{O}_{n,m}\) defines a zeros matrix of dimensions \(n \times m\).

A matrix \(\mathbb{I}_{n}\) defines the identity matrix of dimensions \(n \times n\).

A matrix \(\mathbb{I}_{n,m}\) defines the matrix of dimensions \(n \times m\) having all elements equal to 1.

II. PROBLEM FORMULATION

Consider the following discrete-time LTI (Linear Time Invariant) system:
\[
\begin{align*}
\begin{cases}
x_{k+1} &= A G_{1c} x_k + B H_{1a} u_k + E \omega_k \\
y_k &= C I_{1c} x_k + F \omega_k
\end{cases}
\end{align*}
\]
with \(x \in \mathbb{R}^{n_x} , B \in \mathbb{R}^{n_x \times n_u} , C \in \mathbb{R}^{n_y \times n_x} , E \in \mathbb{R}^{n_y \times (n_1 + n_2 + n_3)} , G_{1c} \in \mathbb{R}^{n_x \times n_x} , H_{1a} \in \mathbb{R}^{n_y \times n_x} \) and \( I_{1c} \in \mathbb{R}^{n_x \times n_x} . \) \( x_k \in \mathbb{R}^{n_x} \) is the state vector of the system, \( u_k \in \mathbb{R}^{n_u} \) is the input vector, and \( y_k \in \mathbb{R}^{n_y} \) is the measured output vector at sample time \( k \). The vector \( \omega_k \in \mathbb{R}^{n_1 + n_2 + n_3} \) contains the state perturbations and the measurement perturbations (noise, offset, etc.). The perturbations are assumed to be bounded by unitary boxes \( \omega_k \in \mathbb{B}^{n_1 + n_2 + n_3} \) for every \( k \geq 0 \). Consider that the initial state \( x_0 \) belongs to the ellipsoid \( E(P_0, x_0, \rho_0) = \{ x \in \mathbb{R}^{n_x} : (x - \bar{x}_0)^T P_0 (x - \bar{x}_0) \leq \rho_0 \} \).

The matrix \( G_{1c} \), with \( i_c \in I_c = \{ 0, 1, 2, \ldots, n_c \} \) and \( n_c \) denoting the number of the considered component faults, is a diagonal matrix modeling the \( i_c \)-th component mode. In a similar way, the matrix \( H_{1a} \), with \( i_a \in I_a = \{ 0, 1, 2, \ldots, n_a \} \) and \( n_a \) the number of considered actuator faults, is a diagonal matrix modeling the \( i_a \)-th actuator mode. The matrix \( I_{1c} \), with \( i_s \in I_s = \{ 0, 1, 2, \ldots, n_s \} \), where \( n_s \) denotes the number of considered sensor faults, is a diagonal matrix modeling the \( i_s \)-th sensor mode.

All diagonal entries of \( G_{1c} , H_{1a} \) and \( I_{1c} \) belong to \([0, 1]\) where 0 or 1 means that the corresponding components, actuators and sensors are completely faulty or healthy, respectively. A value in the range \((0, 1)\) denotes a partial degradation of the corresponding components, actuators and sensors.

It is assumed that the pairs \((A G_{1c}, BH_{1a})\) and \((A G_{1c}, CI_{1c})\) are respectively stabilizable and detectable under all the considered modes.

Remark 1: The system (4) can be rewritten in the following form:
\[
\begin{align*}
\begin{cases}
x_{k+1} &= A(x_k + f_{x_k}) + B(u_k + f_{u_k}) + E \omega_k \\
y_k &= C x_k + F \omega_k + f_{y_k}
\end{cases}
\end{align*}
\]
where \(f_{x_k}, f_{u_k}\) and \(f_{y_k}\) are respectively the component fault, actuator fault and the sensor fault. It is easy to verify this, by taking \(f_{x_k} = (G_{1c} - \mathbb{I}_{n_x}) x_k, f_{u_k} = (H_{1a} - \mathbb{I}_{n_u}) u_k\) and \(f_{y_k} = (I_{1c} - \mathbb{I}_{n_s}) x_k\).

Given an ellipsoidal estimation for \(x_k\) of the form \(E(P, x, \rho, k)\), with \(P\) unknown and \(k > 0\), the objective of this paper is to provide an ellipsoidal estimation for \(x_{k+1}\) of the form \(E(P, \bar{x}_{k+1}, \rho, k+1)\) using the ellipsoidal set-membership state estimation presented in [BSA+14] despite the presence of possible faults (on components, actuators or sensors).

The next section summarizes the ellipsoidal state estimation technique [BSA+14] used for the fault-free case.

III. ELLIPSOIDAL STATE ESTIMATION OF THE FAULT-FREE SYSTEM

This subsection briefly describes the guaranteed ellipsoidal state estimation [BSA+14] for the system (4) in the fault-free case (i.e. \(G_{1c} , H_{1a} \) and \( I_{1c} \) are identity matrices). In this case, the system (4) becomes:
\[
\begin{align*}
\begin{cases}
x_{k+1} &= A x_k + B u_k + E \omega_k \\
y_k &= C x_k + F \omega_k
\end{cases}
\end{align*}
\]

The ellipsoidal estimation method is based on the minimization of the ellipsoidal radius at each iteration by solving a Linear Matrix Inequality (LMI) problem.

Consider an initial state vector \(x_0 \in E(P_0, \bar{x}_0, \rho_0)\) and assume that \(x_k \in E(P, \bar{x}_k, \rho_k)\) at time \( k \). If there exist a matrix \( Y_k \in \mathbb{R}^{n_x \times n_x} \), a matrix \( S = S^T > 0 \in \mathbb{R}^{(n_1 + n_2 + n_3)(n_1 + n_2 + n_3)}\) and the scalars \(\rho_{k+1} > 0\) and \(\beta \in (0, 1)\) for which the
following LMI holds:

\[
\begin{align*}
&\min_{\beta,Y_k,S,\rho_k+1} \rho_{k+1} \\
&\text{subject to} \\
&\begin{bmatrix}
\beta P & * & * \\
PA - Y_k C & P & * \\
0 & E^T P - F^T Y_k^T & S \\
\end{bmatrix} \succ 0, \\
&\rho_{k+1} - \beta \rho_k > 0, \\
&\beta < 1
\end{align*}
\]  

(7)

then the system state \(x_{k+1}\) at time \(k + 1\) is guaranteed to belong to the ellipsoid \(\mathcal{E}(P, \bar{x}_{k+1}, \rho_{k+1}), \forall \omega_k \in \mathbb{B}^{n_x + n_y + n_s}, \)

with the following notations:

\[
\begin{align*}
Y_k &= PL_k, \\
\bar{x}_{k+1} &= Ax_k + Bu_k + L_k(y_k - C\bar{x}_k).
\end{align*}
\]

(8)

(9)

The proof of this result is given in [BSA+14]. This method allows us to estimate the state of the system (4) in the fault-free case, offering a good trade-off between accuracy and computation time of the estimation.

**Remark 2:** At time \(k = 1\), an initialization step is effectuated. Thus, given a scalar \(\beta \in (0, 1)\), the shape matrix \(P\) is found by solving the following LMI problem (obtained from (7) with \(P\) as decision variable):

\[
\begin{align*}
&\min_{P,Y_k,S,\rho_k+1} \rho_{k+1} \\
&\text{subject to} \\
&\begin{bmatrix}
\beta P & * & * \\
PA - Y_k C & P & * \\
0 & E^T P - F^T Y_k^T & S \\
\end{bmatrix} \succ 0, \\
&\rho_{k+1} - \beta \rho_k > 0,
\end{align*}
\]

Once the matrix \(P\) is fixed, the LMI problem (7) is solved at each time instant.

The objective is twofold:

- Find the models which are compatible with the set of the measurements;

- Use this ellipsoidal estimation method to estimate the state of the system (4) despite the presence of faults.

This idea will be developed in the next subsection.

**IV. MULTIPLE MODELS FAULT DETECTION**

The idea is to construct a set of \(p\) Multiple Models \(\mathcal{M} = \{M_1, M_2, \ldots, M_p\}\) such that \(M_1\) represents the fault-free case, i.e. \(A_1 = A, B_1 = B, C_1 = C, E_1 = E\) and \(F_1 = F.\) For for \(i = 2, \ldots, p\), each model \(M_i\) is dedicated to one faulty mode. Note that the model \(M_i\) is defined by the matrices \(A_i, B_i, C_i, E_i\) and \(F_i.\)

The state of the system (4) is estimated by each model \(M_i\) based on the ellipsoidal estimation (7) presented in the previous section. Considering the presence of faults, the consistency between the model \(M_i\) and the measurement has to be checked at each sample time. Then, the objective is to find the models which are compatible with the set of measurements. Once this set is computed, a Min-Max Model Predictive Control is developed in order to stabilize the state \(x_k\) of the system (4) and to decide which is the best model to estimate the state of the system for the next step. The details of the Min-Max MPC problem are given in Section V.

Algorithm 1 provides a general form of the Fault Detection and Fault Tolerant Control strategy based on checking consistency between the models and the measurements. The idea of this algorithm is summarized below:

- **Initialization:** (step 1 to step 5)

  The state estimation for is initialized by the ellipsoidal set \(\mathcal{E}(P_0, \bar{x}_0, \rho_0)\) in step 2. The estimation for each model \(M_i \in \mathcal{M}, i = 1, \ldots, p,\) is also initialized by the same ellipsoidal set \(\mathcal{E}(P_0, \bar{x}_0, \rho_0) = \mathcal{E}(P_0, \bar{x}_0, \rho_0).\)

- **Compatible models set construction:** (step 7 to step 18)

  At each sample time \(k,\) the output measurement \(y_k\) in (4) obtained from the sensors is used to build the parametrized polytope \(\mathcal{P}_{\text{check}}(C_i, y_k, F_i).\) This polytope corresponds to the consistent state set with the measurements \(y_k\) and the construction of this polytope is detailed in Appendix. The consistency between the ellipsoidal estimated set \(\mathcal{E}_i(P, \bar{x}_i, \rho_i)\) and the polytope \(\mathcal{P}_{\text{check}}(C_i, y_k, F_i)\) is verified for each model \(M_i \in \mathcal{M}.\) The ellipsoidal set \(\mathcal{E}_i(P, \bar{x}_i, \rho_i)\) represents the state estimation with the model \(M_i.\)

  This consistency test is solved by the following Quadratic Programming (QP) optimization problem with linear constraints:

\[
\begin{align*}
t^* &= \min_{x_k \in \mathcal{E}_i(P, \bar{x}_i, \rho_i)} (x_k - \bar{x}_k)^T P (x_k - \bar{x}_k) \\
&\text{subject to} \\
&S x_k \leq T.
\end{align*}
\]

If \(t^* < \rho_k,\) then the intersection \(\mathcal{E}_i(P, \bar{x}_i, \rho_i) \cap \mathcal{P}_{\text{check}}(C_i, y_k, F_i)\) is not empty. Else, the intersection is empty \(\mathcal{E}_i(P, \bar{x}_i, \rho_i) \cap \mathcal{P}_{\text{check}}(C_i, y_k, F_i) = \emptyset.\)

If the consistency is proved (i.e. non-empty intersection), the model \(M_i\) is called **compatible with the measurements** and it is added to the set \(\mathcal{C}_M\) containing all the compatible models with the measurements. Otherwise, the model \(M_i\) is called **incompatible with the measurements.**

- **Construction of a Min-Max Model Predictive Control for each compatible model:** (step 19 to step 24)

  A sequence of control \(u_{k|k} = [u_{k|k,j}, u_{k+1|k,j}, \ldots, u_{k+h-1|k,j}]^T\) is computed for each model \(M_j \in \mathcal{C}_M, j = 1, \ldots, s_M (s_M\) is the size of \(\mathcal{C}_M\),) by minimizing the following criterion:

\[
\begin{align*}
u_{k|k,j} &= \arg \min_{u_{k|k}, \omega_k \in \mathbb{B}^{n_y + n_y}} J_f(u_{k|k,j}, \omega_k|k,j, x_{k|k,j}) \\
&\text{subject to} \\
x_{k+l|k} &= M_j x_{k+l|k} \iff l = 1, \ldots, h \\
u_{k+l|k} &\in U \text{ for } l = 1, \ldots, h
\end{align*}
\]

where \(h\) is the prediction horizon, \(x_{k+l|k}\) represents the prediction of the state for the sample time \(k + l\) at the sample time \(k, u_{k+l|k}\) is the control prediction for the sample time \(k + l\) at the sample time \(k, \omega_{k+l|k}\) is the
perturbation prediction for the sample time $k+l$ at the sample time $k$ and the cost function is defined as:

$$ J_J(u_{k|k}, \omega_{k|k}, x_{k|k}) = $$

$$ \sum_{l=0}^{h-1} \left( x_{k+l|k}^T Q x_{k+l|k} + u_{k+l|k}^T R u_{k+l|k} \right). $$

(13)

The cost function $J_J(u_{k|k}, \omega_{k|k}, x_{k|k})$ is maximized with respect to $\omega_{k+l|k} \in \mathbb{B}^{n_x+ny}$ (corresponding to the worst case situation) and minimized with respect to $u_{k+l|k}$. The index $j$ refers to the model $M_j \in C_M$. Generally, the constraints on the state and input vectors and the choice of the weighting matrices $Q$ and $R$ are due to physical, safety or performance considerations.

Then, the set of controllers $U_k = \{u_{k|k,1}, \ldots, u_{k|k,s_M} \}$ suitable for each model $M_j \in C_M$ is constructed.

- **Computing the optimal control and the best model for the estimation**: (step 25)

The objective is to determine the best control $u^*_{k|k,j} \in U_k$ for the system (4) and the best model $M^*_{j} \in C_M$ to use for the estimation. For this, the following optimization problem is solved:

$$ (u^*_{k|k}, M^*_{j}) = \arg\min_{u_{k|k} \in U_k} \max_{M_j \in C_M} J(u_{k|k}, \omega_{k|k}, x_{k|k}), $$

(14)

with the cost function

$$ J(u_{k|k}, \omega_{k|k}, x_{k|k}) = $$

$$ \sum_{l=0}^{h-1} \left( x_{k+l|k}^T Q x_{k+l|k} + u_{k+l|k}^T R u_{k+l|k} \right). $$

(13)

Based on the receding horizon strategy, the control $u^*_{k|k}$ that will be applied to the system (4) is given by the first $n_u$ components of the control sequence $u^*_{k|k}$ as follows:

$$ u^*_{k|k} = \left[ I_{n_u} \ 0_{n_u \times (N_{pred} - 1)n_u} \right] u^*_{k|k}. $$

(15)

- **Computing the estimation for each model**: (step 26 to step 32)

It consists in computing the ellipsoidal estimated sets $E_i(P, \vec{x}_{k+1}, \rho_{k+1,i})$ for each model $M_i \in M$, for $i = 1, \ldots, p$. If the model $M_i$ was compatible with the measurement $y_k$ (i.e. $M_i \in C_M$), then the ellipsoidal estimation set $E_i(P, \vec{x}_{k+1}, \rho_{k+1,i})$ is computed according to (7) using the model $M_i$, the control $u^*_{k|k}$ and the measurement $y_k$. Otherwise, the ellipsoidal estimation set $E_i(P, \vec{x}_{k+1}, \rho_{k+1,i})$ is computed according to (7) using the best model $M^*_{k}$, the control $u^*_{k|k}$ and the measurement $y_k$.

- **Obtained estimation**: (step 33)

Finally, the ellipsoidal estimation set $E(P, \vec{x}_{k+1}, \rho_{k+1})$ is based on the best model $M^*_{k}$, the optimal control $u^*_{k|k}$ and $y_k$.

**Algorithm 1. Fault Detection**

1. $k \leftarrow 0$;
2. $E(P_0, \bar{x}_0, \rho_0) \leftarrow \{ x \in \mathbb{R}^{n_x} : (x-\bar{x}_0)^T P_0 (x-\bar{x}_0) \leq \rho_0 \}$;
3. for $i = 1 : p$
4.  $E_i(P_0, \bar{x}_0, \rho_0) = E(P_0, \bar{x}_0, \rho_0)$;
5. end for

V. MIN-MAX MODEL PREDICTIVE CONTROL

This section details the development of Min-Max Model Predictive Control applied to each model $M_j$ belonging to the compatible set $C_M$. The control signal is found by minimizing a worst case (with respect to the perturbations $\omega_k$) of a quadratic criterion (12). The Min-Max optimization problem (12) is reformulated as a quadratic programming (QP) problem. The controller is, then, computed using the ellipsoidal state estimation at the previous sample time by solving a simple QP problem.

Starting from the quadratic cost function:

$$ J_j(u_{k,j}, \omega_{k,j}, x_{k,j}) = \sum_{l=0}^{h-1} \left( x_{k+l,j}^T Q x_{k+l,j} + u_{k+l,j}^T R u_{k+l,j} \right) $$

(16)
the following state equations\footnote{Here the index } are computed for each compatible model $M_j \in C_M$:

$$
\begin{aligned}
& x_{k+1,j} = A_j x_{k,j} + B_j u_{k,j} + E_j \omega_{k,j}, \\
& \vdots \\
& x_{k+l,j} = A_j^l x_{k,j} + A_j^{l-1} B_j u_{k,j} + A_j^{l-2} B_j u_{k+1,j} + \\
& \quad + \ldots + B_j u_{k+l-1,j} + A_j^{l-1} F_j \omega_{k,j} + \\
& \quad + A_j^{l-2} F_j \omega_{k+1,j} + \ldots + F_j \omega_{k+l-1,j}, \\
& x_{k+h,j} = A_h^l x_{k,j} + A_h^{l-1} B_j u_{k,j} + A_h^{l-2} B_j u_{k+1,j} + \\
& \quad + \ldots + B_j u_{k+h-1,j} + A_h^{l-1} F_j \omega_{k,j} + \\
& \quad + A_h^{l-2} F_j \omega_{k+1,j} + \ldots + F_j \omega_{k+h-1,j},
\end{aligned}
$$

with $h$ the prediction horizon and $j = 1, \ldots, s_M$. Denote by $u_{k,j} = [u_{k,j}, u_{k+1,j}, \ldots, u_{k+h-1,j}]^T$ and $\omega_{k,j} = [\omega_{k,j}, \omega_{k+1,j}, \ldots, \omega_{k+h-1,j}]^T$ the sequences of control signals and perturbations, respectively. Then, the state equation predicted for time $k+l$ at time $k$ of the model $M_j \in C_M$ can be rewritten as:

$$
\begin{aligned}
x_{k+l|k,j} = A_{l,j} x_{k,j} + A_{l,j} B_j u_{k,j} + A_{l,j} \omega_{k,j}
\end{aligned}
$$

(17)

where the $A_{l,j}$ matrix is defined as:

$$
A_{l,j} = [A_j^{l-1} A_j^{l-2} \ldots A_j] Z_l
$$

with $Z_l = \begin{bmatrix} O_{n_x,n_x} & \ldots & O_{n_x,n_x} \end{bmatrix}$.

Replacing (17) in (16) and after some manipulations, the optimization problem (12) becomes:

$$
\begin{aligned}
& u_{k,j} = \arg \min_{u_{k,j} \in U_k} \max_{\omega_{k,j} \in \mathcal{E}(0,1)^{h\times(n_x+n_y)}} f(u_{k,j}, \omega_{k,j}),
\end{aligned}
$$

(18)

where $f(u_{k,j}, \omega_{k,j}) = \alpha_1 + \alpha_2 \omega_{k,j} + \alpha_3 u_{k,j} + \omega_{k,j}^T A_{l,j} \omega_{k,j} + \alpha_4 u_{k,j} + \alpha_5 u_{k,j} + \alpha_6 u_{k,j}$ with

$$
\begin{aligned}
& \alpha_1 = \sum_{l=0}^{h-1} A_j^T Q A_j \sum_{l=0}^{h-1} A_j^T Q A_j, \\
& \alpha_2 = 2 \sum_{l=0}^{h-1} A_j^T Q A_j \sum_{l=0}^{h-1} A_j^T Q A_j, \\
& \alpha_3 = 2 \sum_{l=0}^{h-1} A_j^T Q A_j \sum_{l=0}^{h-1} A_j^T Q A_j, \\
& \alpha_4 = F^T \sum_{l=0}^{h-1} A_j^T Q A_j \sum_{l=0}^{h-1} A_j^T Q A_j, \\
& \alpha_5 = 2 F^T \sum_{l=0}^{h-1} A_j^T Q A_j \sum_{l=0}^{h-1} A_j^T Q A_j, \\
& \alpha_6 = F^T \sum_{l=0}^{h-1} A_j^T Q A_j \sum_{l=0}^{h-1} A_j^T Q A_j.
\end{aligned}
$$

(19)

The function $f(u_{k,j}, \omega_{k,j})$ is quadratic with respect to $u_{k,j}$ and $\omega_{k,j}$. In [ARdIF05], it is shown that the Min-Max MPC problem (18) is equivalent to:

$$
\begin{aligned}
& u_{k,j} = \arg \min_{u_{k,j} \in U_k} \max_{\omega_{k,j} \in \mathcal{E}(0,1)^{h\times(n_x+n_y)}} f(u_{k,j}, \omega_{k,j}).
\end{aligned}
$$

(20)

The problem (20) becomes a QP problem as follows:

$$
\begin{aligned}
& u_{k,j} = \arg \min_{u_{k,j} \in U_k} \tilde{f}(u_{k,j}),
\end{aligned}
$$

(21)

such that $\tilde{f}(u_{k,j})$ is quadratic with respect to $u_{k,j}$. In general the constraints $x_k \in \mathbb{R}$ and $u_k \in \mathbb{R}$ are given in the following form: $x_{\min} \leq x_k \leq x_{\max}$ and $u_{\min} \leq u_k \leq u_{\max}$.

Finally, the problem (12) to solve is a QP problem:

$$
\begin{aligned}
& \min \tilde{f}(u_{k,j}) \\
& \text{subject to } \begin{bmatrix} A_{l,j} \bar{B} \\ -A_{l,j} \bar{B} \\ \bar{I} \\ -\bar{I} \end{bmatrix} u_{k,j} \leq \begin{bmatrix} b_1 \\ b_2 \\ u_{\max} \\ -u_{\min} \end{bmatrix},
\end{aligned}
$$

(22)

for $l = 1, \ldots, h$ with $b_1 = x_{\max} - A_{l,j} x_k - A_{l,j} \bar{F} \omega_k$ and $b_2 = -x_{\min} + A_{l,j} x_k + A_{l,j} \bar{F} \omega_k$, $\omega_k \in \mathcal{E}(0,1)^{h\times(n_x+n_y)}$ and $\bar{I} = \begin{bmatrix} \bar{I}_{l-1,n_u} & \bar{I}_{l,n_u} & \bar{I}_{h-l,n_u} \end{bmatrix}$.

**Remark 3:** The state $x_{k|k}$ is chosen equal to the nominal state which is in this case the center of the ellipsoidal state estimation set $\bar{x}_k$.

In the next section, an illustrative example showing the effectiveness of the Algorithm I is presented.

VI. ILLUSTRATIVE EXAMPLE

Consider the following LTI discrete-time system:

$$
\begin{aligned}
x_{k+1} = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} x_k + \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} u_k + \\
+ \begin{bmatrix} 0.05 & 0 & 0 & 0 \\ 0 & 0.02 & 0 & 0 \end{bmatrix} \omega_k \\
y_k = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix} \omega_k
\end{aligned}
$$

(23)

with $\|\omega_k\|_\infty \leq 1$. The value of $\omega_k$ is randomly generated. The initial state belongs to the ellipsoid $\mathcal{E}(l_2, [0 \ 0]^T, 1)$. In this example, 4 models are considered. $M_1$ corresponds to the fault-free system, i.e. $A_1 = A$, $B_1 = B$, $C_1 = C$, $E_1 = E$ and $F_1 = F$. $M_2$ models the system with a component fault: $A_2 = \begin{bmatrix} 0.4 & 0.8 \\ 0.1 & 0.2 \end{bmatrix}$, $B_2 = B$, $C_2 = C$, $E_2 = E$ and $F_2 = F$. $M_3$ corresponds to an actuator fault, with $A_3 = A$, $B_2 = \begin{bmatrix} 0.15 & 0.1 \\ 0 & 1 \end{bmatrix}$, $C_3 = C$, $E_3 = E$ and $F_3 = F$. $M_4$ corresponds to the system having a partial fault in the second sensor: $A_4 = A$, $B_4 = B$, $C_4 = \begin{bmatrix} -2 & 1 \\ 0.5 & 0.5 \end{bmatrix}$, $E_4 = E$ and $F_4 = F$. The length simulation $N = 100$. The prediction horizon $h = 10$, the weighting matrices $Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ and $R = 5$. The following constraints are considered on
the state $x_{\text{min}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $x_{\text{max}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and on the input $u_{\text{min}} = -0.8$ and $u_{\text{max}} = 0.8$.

The simulated faults are described in Table I.

<table>
<thead>
<tr>
<th>Fault description</th>
<th>Time interval (samples)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50% fault in actuator 1</td>
<td>10 – 20</td>
</tr>
<tr>
<td>50% fault in sensor</td>
<td>50 – 60</td>
</tr>
</tbody>
</table>

Figures 1, 2 illustrate the bounds of $x_1$ and $x_2$ after 100 iterations. The solid blue lines represent the bounds obtained by Algorithm 1. The red stars represent the real state of the system (situated inside the estimated bounds). The state estimation is guaranteed despite the presence of the considered faults, however the bounds of the estimation set are larger when faults occur (compared to a fault-free time intervals).

Figures 3, 4, 5, 6, 7 represent the fault signal obtained by models $M_1$, $M_2$, $M_3$ and $M_4$. When the fault signal is equal to 0 (respectively 1), the model $M_i$ is compatible (respectively incompatible) with the measurements. Effectively, the model $M_1$ corresponding to the fault-free case system is compatible with the measurement when there is no fault. Even if for the considered actuator fault (between 10 – 20 samples), the models $M_2$, $M_3$ and $M_4$ are compatible with the measurements, the optimal model chosen by the Min-Max MPC is $M_3$. In a similar way, the model $M_4$ is the optimal model for the considered sensor fault. This confirms the performance of Algorithm 1.

Figure 3 represents the control $u_k$. The constraint $u_{\text{min}} \leq u_k \leq u_{\text{max}}$ is satisfied.

VII. CONCLUSION

A new Fault Detection algorithm based on Multiple Models for linear systems with bounded perturbations and measurement noises has been proposed. The proposed algorithm allows to estimate the state of the system despite the presence of faults. A Min-Max MPC based on the ellipsoidal
Fig. 5. Fault signal for model $M_2$

Fig. 6. Fault signal for model $M_3$

Fig. 7. Fault signal for model $M_4$

state estimation has been used. An example illustrates the effectiveness of the proposed method.

An interesting perspective is to extend this to the case of systems with interval uncertainties.

**APPENDIX**

This part details the construction of the polytope $\mathcal{P}_{\text{check}}(C_i, y_k, F_i)$, obtained from the intersection of all the $n_y$ measurement strips. Each strip is defined by these two inequalities:

\[
\begin{align*}
C_{i,j} & \leq y_{k,j} + \|F_{i,j}\|_1 \\
-C_{i,j} & \leq -y_{k,j} + \|F_{i,j}\|_1
\end{align*}
\]

such that $i$ represents the $i^{th}$ model and $j$ represents the $j^{th}$ line of $C_i$, $F_i$ and $y_k$.

Then, the polytope $\mathcal{P}_{\text{check}}(C_i, y_k, F_i)$ is defined by the following constraints:

\[\mathcal{P}_{\text{check}}(C_i, y_k, F_i) = \{x \in \mathbb{R}^{n_x} : Sx_k \leq T\},\]

with the matrices $S = \begin{bmatrix} C_i \\ -C_i \end{bmatrix}$, $T = \begin{bmatrix} y_k + F_i \\ -y_k + F_i \end{bmatrix}$ and $F_i = \begin{bmatrix} \|F_{i,1}\|_1 \\ \vdots \\ \|F_{i,n_y}\|_1 \end{bmatrix}$. Note that $F_{i,j}$ designs the $j^{th}$ line of the $F_i$ matrix for the model $M_i$.

**REFERENCES**


