Rectified ALS Algorithm for Multidimensional Harmonic Retrieval

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***Abstract*—MultiDimensional (MD) Harmonic Retrieval is a challenging multi-parameter estimation problem and is useful for a plethora of operational applications as for instance channel sounding or MIMO radar processing. The MD-harmonic model follows a structured Canonical Polyadic Decomposition (CPD) in the sense that the factors of the CPD are Vandermonde. A standard and popular estimation scheme to derive the CPD is the Alternating Least Squares (ALS) algorithm. Unfortunately, the ALS algorithm does not exploit the a priori known factor structure, which considerably degrades the estimation performance. In this work, a modified ALS-type algorithm is proposed. This new algorithm, called Rectified ALS (RecALS), is able to take into account the Vandermonde structure of the factors. The RecALS algorithm belongs to the Lift-and-Project family and exploits iterated projections on the set of Toeplitz rank-1 matrices. It exhibits a fast convergence and is very accurate in the sense that its Mean Square Error (MSE) is close to the Cramér-Rao Bound for a wide range of Signal to Noise Ratio (SNR).**

***Index Terms*—Structured Canonical Polyadic Decomposition, Vandermonde factors, MultiDimensional Harmonic Retrieval, modified Alternating Least Squares algorithm, Toeplitz rank-1 matrix approximation**

I. Introduction

An increasing number of signal processing applications deal with multidimensional data. In particular, MultiDimensional (MD)-harmonic retrieval [1]–[4] is an important and challenging multi-parameter estimation problem. Multilinear algebra provides a powerful framework to exploit these data [5]–[7] by conserving the multidimensional structure of the information.

Nevertheless, generalizing matrix-based algorithms to the multilinear algebra framework is not a trivial task. In particular, there exist several multilinear extensions of the matrix Singular Value Decomposition (SVD) [8], each enjoying only some properties of the matrix SVD. The Canonical Polyadic Decomposition (CPD), also sometimes referred to as Candecomp/Parafac may be seen as one possible extension of the SVD to the multilinear case; see [6] and references therein. Recall that a rank-M matrix is defined as the sum of M rank-1 matrices and the best low-rank approximation is provided by the truncated-SVD [25]. The CPD straightforwardly extends this principle, *i.e.*, a rank-M tensor follows a CPD of order M or equivalently can be expressed as the sum of M rank-1 tensors, which can be stored in matrix factors. In addition, like SVD, it is essentially unique under mild conditions. Unfortunately, in contrast to matrices, the set of tensors of rank at most M is in general not closed, which renders the problem of finding a tensorial best low-rank approximation ill-posed [27]. As a consequence, we have to resort to sub-optimal algorithms to derive the CPD.

Despite of this difficulty, the Alternating Least Squares (ALS) algorithm [7,9,10] has proven to be powerful in a wide range of operational contexts. To such an extent that the ALS algorithm is now considered as the gold standard method to estimate the factors entering the CPD. However, in the standard ALS algorithm, all factor entries are estimated independently while ignoring an *a priori* known structure. In addition, practical problems are encountered when factor matrices have a linear structure [11,12] such as Toeplitz, circulant or Hankel. In order to fill this gap, several algorithms have been proposed and their estimation accuracies have been studied in [13]. But less attention has been dedicated to Vandermonde structured factors [1]–[3] involved in the CPD of the MD-harmonic model.

A class of well-known methods called projections onto convex sets (POCS) continues to receive great attention [15,17,24,26]. The POCS method is based on an infinite sequence of iterated projections onto closed convex subsets. This method is simple to implement and its convergence has been proved in [28] for instance. Unfortunately, in many operational applications, the considered subsets are nonconvex and generally no convergence guarantees exist. A typical example of nonconvex problem is the well-known phase recovery problem [29]. Despite this, a method called Lift-and-Project has been the subject of many publications [16] and has been used successfully in several operational contexts [18,23]. In this work, we propose a new methodology belonging to the Lift-and-Project algorithm family to tackle the MD-Harmonic Retrieval problem, breaking with the standard approaches as proposed for instance in [14].

II. MD-harmonic model and ALS algorithm

A. Structured Canonical Polyadic Decomposition (CPD)

The MD-harmonic model assumes that the measurements can be modeled as the superposition of M undamped exponentials sampled on a P-dimensional grid according to

$$[\boldsymbol{\mathcal{X}}]_{n_1\ldots n_P} = \sum_{m=1}^{M} \alpha_m \prod_{p=1}^{P} z_{m,p}^{n_p}, \quad 0 \le n_p \le N_p - 1 \qquad (1)$$

in which the m-th complex amplitude is denoted by α_m and $z_{m,p} = e^{i\omega_{m,p}}$ where $\omega_{m,p}$ is the m-th angular-frequency along the p-th dimension. Tensor $\boldsymbol{\mathcal{X}}$ can be expressed as the linear combination of M rank-1 tensors, each of size $N_1 \times \ldots \times N_P$ (the size of the grid), and follows a generalized Vandermonde decomposition:

$$\boldsymbol{\mathcal{X}} = (\mathbf{V}_1, \ldots, \mathbf{V}_P) \cdot \boldsymbol{\mathcal{A}}$$

where $\boldsymbol{\mathcal{A}}$ is a $M \times \ldots \times M$ diagonal tensor with $[\boldsymbol{\mathcal{A}}]_{m,\ldots,m} = \alpha_m$ and $\mathbf{V}_p = \begin{bmatrix}\mathbf{v}(z_{1,p}) & \ldots & \mathbf{v}(z_{M,p})\end{bmatrix}$ is a $N_p \times M$ Vandermonde matrix, where each column $\mathbf{v}(z_{m,p})$ depends on a single parameter, $z_{m,p}$, of unit modulus. We define a noisy MD-harmonic tensor model of order P as: $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} + \sigma\boldsymbol{\mathcal{E}}$ where $\sigma\boldsymbol{\mathcal{E}}$ is the noise tensor, σ is a positive real scalar, and each entry $[\boldsymbol{\mathcal{E}}]_{n_1\ldots n_P}$ follows an i.i.d. circular complex Gaussian distribution $\mathcal{N}(0,1)$, and $\boldsymbol{\mathcal{X}}$ has rank M.

B. Limit of the ALS algorithm for structured CPD

The CPD of any order-P rank-M tensor $\boldsymbol{\mathcal{X}}$ involves the estimation of P factors $\mathbf{F}_p$ of size $N_p \times M$. As pointed out above, in the context of the MD-harmonic model, the factors $\mathbf{F}_p$ of the CPD are Vandermonde matrices. Consider, $\mathbf{Y}_p$, the p-th mode unfolding [6] of tensor $\boldsymbol{\mathcal{Y}}$, at the k-th iteration with $1 \le k \le I$, I denoting the maximal number of iterations. The ALS algorithm solves alternatively for each of the P dimensions the minimization problem [7,9]: $\min_{\mathbf{F}_p} ||\mathbf{Y}_p - \mathbf{F}_p\mathbf{G}_p||^2$ where $\mathbf{G}_p^T = \mathbf{F}_1 \odot \ldots \odot \mathbf{F}_{p-1} \odot \mathbf{F}_{p+1} \odot \ldots \odot \mathbf{F}_P$, where $\odot$ denotes the Khatri-Rao product [6]. It aims at approximating tensor $\boldsymbol{\mathcal{Y}}$ by a tensor of rank M, hopefully close to $\boldsymbol{\mathcal{X}}$. The LS solution conditionally to matrix $\mathbf{G}_p$ is given by $\mathbf{F}_p = \mathbf{Y}_p\mathbf{G}_p^{\dagger}$ where $\dagger$ stands for the pseudo-inverse. Now, remark that there is no reason that the above LS criterion promotes the Vandermonde structure in the estimated factors in the presence of noise. In other words, ignoring the structure in the CPD leads to estimate an excessive number of free parameters. This mismatched model dramatically decreases the estimation performance [13]. Hence there is a need to rectify the ALS algorithm to take into account the factor structure.

III. Rectification strategies

A. Iterated projections and splitted LS criterion

Let $\mathbb{V}$ be the set of the complex Vandermonde vectors. The structured LS optimization problem is the following [15]–[17]:

$$\min_{\mathbf{V}_p} \mathrm{Trace}\Big\{(\mathbf{F}_p - \mathbf{V}_p)(\mathbf{F}_p - \mathbf{V}_p)^H\Big\} \text{ s.t. } \mathbf{V}_p \in \mathbb{V}^M \qquad (2)$$

where $\mathbb{V}^M$ is the set of Vandermonde matrices with M columns. The proposed strategy is to split the optimization problem (2) into the resolution of M independent LS subproblems:

$$\min_{\mathbf{v}} ||\mathbf{f}_m^{(p)} - \mathbf{v}||^2 \text{ s.t. } \mathbf{v} \in \mathbb{V} \qquad (3)$$

where $\mathbf{f}_m^{(p)}$ is the m-th column of $\mathbf{F}^{(p)}$.

Assume that set $\mathbb{V}$ is the intersection of K closed convex subsets $\{\mathbb{V}_1 \ldots, \mathbb{V}_K\}$ in which each subset encodes a desired algebraic property. Then the solution of eq. (3) can be rewritten as $\mathbf{v} = \pi_{\mathbb{V}_1 \cap \ldots \cap \mathbb{V}_K}(\mathbf{f}_m^{(p)})$, where $\pi_{\mathbb{V}}$ denotes the projector onto $\mathbb{V}$. A standard approach to solve this optimization problem is the method of iterated projections. More precisely, define the following recursion:

$$\mathbf{f}_m^{(p)}(h) = (\pi_{\mathbb{V}_K} \ldots \pi_{\mathbb{V}_1})(\mathbf{f}_m^{(p)}(h-1)) = (\pi_{\mathbb{V}_K} \ldots \pi_{\mathbb{V}_1})^h(\mathbf{f}_m^{(p)})$$

with $\mathbf{f}_m^{(p)}(0) = \mathbf{f}_m^{(p)}$. Under rather mild conditions, convergence is ensured: [15,16]:

$$\lim_{h\to\infty} ||(\pi_{\mathbb{V}_K} \ldots \pi_{\mathbb{V}_1})^h(\mathbf{f}_m^{(p)}) - \pi_{\mathbb{V}_1 \cap \ldots \cap \mathbb{V}_K}(\mathbf{f}_m^{(p)})||^2 = 0.$$

Unfortunately, many operational applications involve non-convex sets, as for instance the set of rank deficient matrices. In this case, the projection may be multivalued, and there is no guarantee of convergence, even if numerical convergence has often been observed [18].

B. Standard strategies

1) Column-averaging: The most intuitive way to exploit the Vandermonde structure is to note that $\omega = \frac{1}{n}\angle z^n$ where $\angle$ stands for the angle function. Define the sets

$$\mathbb{J} = \left\{\mathbf{v} = \frac{\mathbf{f}}{[\mathbf{f}]_1} : \mathbf{f} \in \mathbb{C}^N\right\}$$

$$\mathbb{A}_\ell = \left\{\mathbf{v}(z = e^{i\bar{\omega}}) : \bar{\omega} = \frac{1}{\ell}\sum_{n=1}^{\ell}\frac{1}{n}\angle[\mathbf{f}]_{n+1}\right\}$$

where $1 \le \ell \le N-2$. This method exploits the Vandermonde structure in a heuristic way. So, the rectified strategy is to consider the iterated vector $\mathbf{f}(h) = (\pi_{\mathbb{A}_\ell}\pi_{\mathbb{J}})^h(\mathbf{f})$.

2) Periodogram maximization: Under Gaussian noise and for a single tone $\alpha_m z_{m,p}^{n_p}$, the maximum likelihood estimator (MLE) is optimal and is given by the location of the maximal peak of the Fourier-periodogram [2,19]. To increase the precision of the estimation, it is standard to use the well-known zero-padding technique at the price of an increase in computational cost.

IV. Rectified ALS (RecALS) algorithm

The RecALS algorithm belongs to the family of Lift-and-Project Algorithms [16,17]. The optional lift step computes a low rank approximation and the projection step performs a rectification toward the desired structure. In Section IV-A the basic RecALS algorithm is described. In Section IV-B, an improved version of the RecALS algorithm is proposed.

A. Principles of the RecALS algorithm

The RecALS algorithm is based on iterated projections and splitted LS criteria. Its algorithmic description is provided in Algorithm 1 for $P = 3$. We insist that several iterations in the while loops are necessary, since restoring the structure generally increases the rank, and computing the low-rank approximation via truncated SVD generally destroys the structure.

Algorithm 1 Rectified ALS (RecALS)

Require: $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, I, \mathbf{F}_1, \mathbf{F}_2, \{\mathbb{V}_1, \dots, \mathbb{V}_K\}$, `CritStop`
Ensure: $\{z_{1,p}, \dots, z_{M,p}\}$ for $1 \leq p \leq 3$

for $k = 1, \dots, I$ **do**
    $\mathbf{F}_3 = \mathbf{Y}_3 \left((\mathbf{F}_1 \odot \mathbf{F}_2)^T\right)^\dagger$
    for $m = 1, \dots, M$ **do**
        $\mathbf{f} := [\mathbf{F}_3]_m$
        while (`CritStop` is false) **do**
            $\mathbf{f} = \pi_{\mathbb{V}_K} \dots \pi_{\mathbb{V}_1}(\mathbf{f})$
        end while
        $z_{m,3} = \min_z ||\mathbf{v}(z) - \mathbf{f}||^2$
    end for
    $\mathbf{F}_3 := \begin{bmatrix}\mathbf{v}(z_{1,3}) & \dots & \mathbf{v}(z_{M,3})\end{bmatrix}$
    $\mathbf{F}_2 = \mathbf{Y}_2 \left((\mathbf{F}_3 \odot \mathbf{F}_1)^T\right)^\dagger$
    for $m = 1, \dots, M$ **do**
        $\mathbf{f} := [\mathbf{F}_2]_m$
        while (`CritStop` is false) **do**
            $\mathbf{f} = \pi_{\mathbb{V}_K} \dots \pi_{\mathbb{V}_1}(\mathbf{f})$
        end while
        $z_{m,2} = \min_z ||\mathbf{v}(z) - \mathbf{f}||^2$
    end for
    $\mathbf{F}_2 := \begin{bmatrix}\mathbf{v}(z_{1,2}) & \dots & \mathbf{v}(z_{M,2})\end{bmatrix}$
    $\mathbf{F}_1 = \mathbf{Y}_1 \left((\mathbf{F}_2 \odot \mathbf{F}_3)^T\right)^\dagger$
    for $m = 1, \dots, M$ **do**
        $\mathbf{f} := [\mathbf{F}_1]_m$
        while (`CritStop` is false) **do**
            $\mathbf{f} = \pi_{\mathbb{V}_K} \dots \pi_{\mathbb{V}_1}(\mathbf{f})$
        end while
        $z_{m,1} = \min_z ||\mathbf{v}(z) - \mathbf{f}||^2$
    end for
    $\mathbf{F}_1 := \begin{bmatrix}\mathbf{v}(z_{1,1}) & \dots & \mathbf{v}(z_{M,1})\end{bmatrix}$
end for

B. Toeplitz Rank-1 Approximation ($\mathrm{TR}_1\mathrm{A}$)

In this section, we propose better strategies than brute-force ALS. This is made possible by noting that columns of Vandermonde matrices can be computed one by one, and that the outer product between a Vandermonde vector and its Hermitian transpose is a rank-one Toeplitz matrix.

1) Equivalent matrix-based criterion:

Property 4.1: Let $\mathbb{T}$ be the set of Hermitian Toeplitz rank-1 matrices of size $N \times N$. For $\mathbf{x} \in \mathbb{R} \times \mathbb{C}^{N-1}$, define the Hermitian Toeplitz matrix:

$$\mathrm{Toep}(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_N \\ x_2^* & x_1 & x_2 & \dots & x_{N-1} \\ x_3^* & x_2^* & x_1 & \dots & x_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_N^* & x_{N-1}^* & x_{N-2}^* & \dots & x_1 \end{bmatrix}.$$

For $\mathbf{v}(z) \in \mathbb{V}$, it is straightforward to prove that:

$$\mathbf{T}(z) = \mathrm{Toep}(\mathbf{v}(z)) = \mathbf{v}(z)^* \mathbf{v}(z)^T \in \mathbb{T}.$$

Property 4.2: Recall that in virtue of the Carathéodory's theorem [20], there exist a one-to-one mapping between $\mathbf{T}(z) \in \mathbb{T}$ and $\mathbf{v}(z) \in \mathbb{V}$ with a unique z. This means that minimizing criteria $||\mathbf{v}(\hat{z}) - \mathbf{v}(z)||^2$ or $||\mathbf{T}(\hat{z}) - \mathbf{T}(z)||^2$ are equivalent.

So, thanks to the two above properties, we propose the following estimation methodology.

Result 4.3: The aim is to find the minimizer, $\hat{z}$, of criterion $||\mathbf{T}(\hat{z}) - \mathbf{T}(z)||^2$ conditionally to $\mathbf{T}(z)$ for unknown z. The solution is given by $\hat{z} = e^{i\angle([\mathbf{u}]_1[\mathbf{u}]_2^*)}$ where $\mathbf{T}(z) = \mathrm{Toep}(\mathbf{u})$.

Proof As matrix $\mathbf{T}(z)$ is by construction Toeplitz and rank-1, we have $\mathbf{T}(z) \stackrel{\mathrm{SVD}}{=} \lambda \mathbf{u}\mathbf{u}^H$. Identifying the (n, n')-th term of the Toeplitz matrix $\mathbf{T}(\hat{z})$ and the (n, n')-th term of the SVD of $\mathbf{T}(z)$ we have $\hat{z}^{-n+n'} = e^{i\omega(-n+n')} = \lambda [\mathbf{u}]_{n+1}[\mathbf{u}]^*_{n'+1}$. So, for $n = 0$ and $n' = 1$, we have $\hat{z} = e^{i\hat{\omega}}$ with $\hat{\omega} = \angle([\mathbf{u}]_1[\mathbf{u}]_2^*)$.

2) Iterated projections: In preamble, the two following sets are introduced.

- Define the set of Hermitian Toeplitz matrices:
$$\mathbb{D} = \left\{\mathbf{T} = \mathrm{Toep}(\mathbf{x}) \in \mathbb{C}^{N \times N}, \, \mathbf{x} \in \mathbb{R} \times \mathbb{C}^{N-1}\right\}.$$
- Introduce the non-convex set of Hermitian rank-1 $N \times N$ matrices:
$$\mathbb{Q} = \left\{\mathbf{Q} \in \mathbb{C}^{N \times N}, \mathrm{rank}\mathbf{Q} = 1\right\}.$$

Clearly, the set $\mathbb{T}$ is nothing else but $\mathbb{Q} \cap \mathbb{D}$. From Property 4.1, we consider the sequence $(\pi_{\mathbb{D}}\pi_{\mathbb{Q}})^h(\mathrm{Toep}(\pi_{\mathbb{J}}\mathbf{f}))$. As $\pi_{\mathbb{J}}\mathbf{f} \notin \mathbb{V}$, $\mathrm{Toep}(\pi_{\mathbb{J}}\mathbf{f})$ is full-rank with probability one. In practice, projector $\pi_{\mathbb{Q}}$ is implemented by just retaining the dominant singular triplet, which is known to yield the optimal rank-1 approximation [8,21]: $\mathbf{Q}^\star = \pi_{\mathbb{Q}}(\mathbf{Q}) = \lambda_{\max}(\mathbf{Q})\mathbf{s}\mathbf{s}^H$ where $\mathbf{s}$ is the singular vector of matrix $\mathbf{Q}$ associated with the largest singular value, $\lambda_{\max}(\mathbf{Q})$.

In the remainder, we shall use the following iterates, with $\mathbf{T} = \mathrm{Toep}(\mathbf{f})$ as initialization:

$$\mathbf{T} \leftarrow \pi_{\mathbb{D}}\pi_{\mathbb{Q}}(\mathbf{T}).$$

Once $(\pi_{\mathbb{D}}\pi_{\mathbb{Q}})^h(\mathbf{T})$ has converged to a matrix in set $\mathbb{T}$, we use Result 4.3 to estimate parameter z.

V. NUMERICAL SIMULATIONS

We fix $M = 2$ sources with $\alpha_1 = e^{i\pi/3}$ and $\alpha_2 = e^{i\pi/4}$ and $P = 3$ in eq. (1). In Fig. 1, the MSE with respect to the SNR is drawn for

1) the RecALS algorithm using the column-averaging defined in section 3.2.1.
2) The RecALS-TR_1A algorithm exploiting iterated projections on the set of Toeplitz rank-1 matrices described in section 4.2.
3) The standard ALS algorithm rectified in a post-processing way with the maximization periodogram procedure, called ALS+$\max$ Perio.
4) The deterministic CRB has been derived in several publications as for instance [4,22] and we use the scalar

expressions given in [3] defined by $\text{Trace}\{\text{CRB}\} = \sum_{p,m} \text{CRB}(\omega_{m,p})$ where $\text{CRB}(\omega_{m,p}) = \frac{6\sigma^2}{N_p^2 \prod_{p \neq p'} N_{p'}}$.

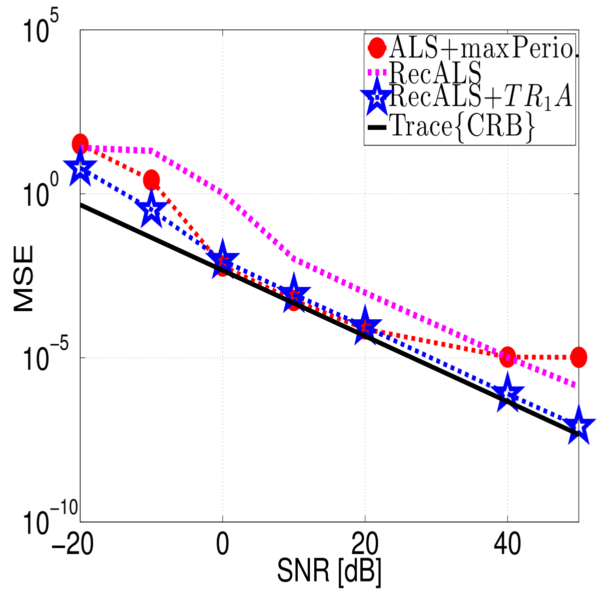

Fig. 1. MSE *Vs.* SNR in dB with $N_1 = N_2 = N_3 = 6$. The zero-padding factor in the ALS+max Perio is 2^{10}.

SNR [dB]	-20	0	10	20	40	50
ALS+max Perio	204.8	48.2	33.5	50.5	44.9	66.9
RecALS	20	19.9	20	20	20	20
RecALS + TR_1A	20	19	19	18.9	19	18.6

TABLE I
MEAN NUMBER OF ITERATIONS *Vs.* SNR IN DB

As shown in Fig. 1, the accuracy of the ALS+max Perio is saturated in the high SNR regime due to the zero-padding processing. Conversely, the two RecALS algorithms do not suffer from this drawback. In addition, according to Table I, the RecALS algorithms converge much faster than the ALS+max Perio. Fig. 2 illustrates the fast convergence of the iterated projections on the set of Toeplitz rank-1 matrices. Finally thanks to Fig. 1, we can observe the RecALS-TR_1A algorithm has the smallest MSE remaining close to the CRB for a wide range of SNR.

VI. Conclusion

MultiDimensional (MD) Harmonic Retrieval is a challenging multi-parameter estimation problem. The MD-harmonic model can be decomposed as a Vandermonde-structured Canonical Polyadic Decomposition (CPD). A popular way to derive the CPD is the Alternating Least Squares (ALS) algorithm. Unfortunately, this scheme ignores the structure in the CPD factors and thus the number of free parameters is overestimated. As a consequence, discarding this a priori knowledge considerably degrades the estimation performance. In this work, we propose a modified ALS-type algorithm, called Rectified ALS (RecALS), which enforces the Vandermonde structure of the CPD factors at each iteration. We

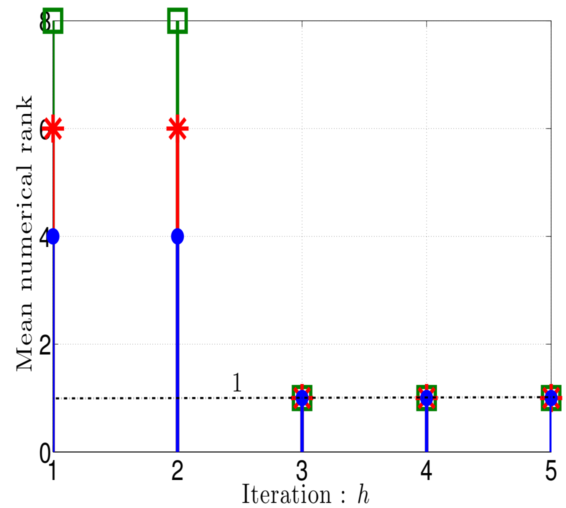

Fig. 2. Mean numerical rank of $(\pi_{\mathbb{Q}}\pi_{\mathbb{D}})^h(\mathbf{f}_1^{(p)})$ with $N_1 = 6$, $N_2 = 8$ and $N_3 = 4$ and SNR = 10 dB

first show that the RecALS algorithm converges much faster than the ALS algorithm. Next, an improved RecALS scheme, based on iterated projections on the set of Toeplitz rank-1 matrices, is introduced. This last scheme is shown to have a fast convergence and its MSE is close to the CRB for a wide range of SNR values.

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