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Bayesian Lower Bounds for Dense or Sparse (Outlier) Noise in the RMT Framework

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Abstract—Robust estimation is an important and timely research subject. In this paper, we investigate performance lower bounds on the mean-square-error (MSE) of any estimator for the Bayesian linear model, corrupted by a noise distributed according to an i.i.d. Student's t-distribution. This class of prior parametrized by its degree of freedom is relevant to modelize either dense or sparse (accounting for outliers) noise. Using the hierarchical Normal-Gamma representation of the Student's t-distribution, the Van Trees' Bayesian Cramér-Rao bounds (BCRBs) on the amplitude parameters and the noise hyper-parameter are derived. Furthermore, the Random Matrix Theory (RMT) framework is assumed, *i.e.*, the number of measurements and the number of unknown parameters grow jointly to infinity with an asymptotic finite ratio. Using some powerful results from the RMT, closed-form expressions of the BCRB are derived and studied. Finally, we propose a framework to fairly compare two models corrupted by noises with different degrees of freedom for a fixed common target signal-to-noise ratio (SNR). In particular, we focus our effort on the comparison of the BCRBs associated with two models corrupted by a sparse noise promoting outliers and a dense (Gaussian) noise, respectively.

Index Terms—Bayesian Hierarchical Linear Model, Bayesian Cramér-Rao bound, sparse outlier noise, dense noise, Random Matrix Theory

I. INTRODUCTION

In the context of robust data modeling [1], the measurement vector may be corrupted by noise outliers. This class of noise is sometimes referred to as sparse noise and is described by a distribution with heavy-tails [2]–[7]. Conversely, we usually call dense a noise that does not share this property and the most popular prior is probably Gaussian noise. Depending on the application context, outliers may be identified, *e.g.*, as incomplete data [8] or corrupted information.

A robust and relevant noise prior which is able to take into account of outliers is the Student's t-distribution for low degrees of freedom [9]–[12]. In addition, dense noise can also be encompassed thanks to the Student's t-distribution prior for an infinite degree of freedom. A convenient framework to deal with a wide class of distributions is well known under the name of hierarchical Bayesian modeling. The Bayesian Hierarchical Linear Model (BHLM) with hierarchical noise prior is used in a wide range of applications, including fusion [13], anomaly detection [5] of hyperspectral images, channel

estimation [14], blind deconvolution [15], segmentation of astronomical times series [16], *etc.*

In this work, we adopt this hierarchical prior framework due to its flexibility and ability to modelize a wide class of priors. More precisely, the noise vector is assumed to follow a circular i.i.d. centered Gaussian prior with a variance defined by the inverse of an unknown random hyper-parameter. In addition, if this hyper-parameter is Gamma distributed [17,18], then the marginalized joint pdf over the hyper-parameter is the Student's t-distribution.

The Van Trees' Bayesian Cramér-Rao bound (BCRB) [19] is a standard and fundamental lower bound on the mean-square-error (MSE) of any estimator. The aim of this work is to derive and analyze the BCRB of the amplitude parameters and the noise hyper-parameter (*i*) for the considered noise prior and (*ii*) using some powerful results from the Random Matrix Theory (RMT) framework [20]–[22]. Regarding reference [23], the proposed work is original in the sense that the noise prior is different and the asymptotic regime is assumed. Finally, note that reference [24] tackles a similar problem but does not assume the asymptotic context.

We use the following notation. Scalars, vectors and matrices are denoted by italic lower-case, boldface lower-case and boldface upper-case symbols, respectively. The symbol $\text{Tr}[\cdot]$ stands for the trace operator. The $K \times K$ identity matrix is denoted by \mathbf{I}_K and $\mathbf{0}_{K \times 1}$ is the $K \times 1$ vector, filled with zeros. The probability density function (pdf) of a given random variable u is given by $p(u)$. The symbol $\mathcal{N}(\cdot, \cdot)$ refers to the Gaussian distribution, parametrized by mean and covariance matrix, $\mathcal{G}(\cdot, \cdot)$ is the Gamma distribution, described by shape and rate (inverse scale) parameters, while $\mathcal{IG}(\cdot, \cdot)$ is the inverse-Gamma distribution. If we have $u \sim \mathcal{G}(a, b)$ then $p(u|a, b) = \frac{b^a u^{a-1} e^{-bu}}{\Gamma(a)}$, where $\Gamma(\cdot)$ is the Gamma function. And if $u \sim \mathcal{IG}(a, b)$, then $p(u|a, b) = \frac{b^a u^{-a-1} e^{-\frac{b}{u}}}{\Gamma(a)}$. The non-standardized Student's t-distribution is defined by three parameters, through the pdf $p(u|\mu, \sigma^2, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu\sigma^2}} (1 + \frac{(u-\mu)^2}{\nu\sigma^2})^{-\frac{\nu+1}{2}}$ such that $u \sim \mathcal{S}(\mu, \sigma^2, \nu)$. As regards the bivariate Normal-Gamma distribution, if we have $(u, w) \sim \text{NormalGamma}(\mu, \lambda, a, b)$, then $p(u, w|\mu, \lambda, a, b) = \frac{b^a \sqrt{\lambda}}{\Gamma(a)\sqrt{2\pi}} w^{a-\frac{1}{2}} e^{-bw} e^{-\frac{\lambda w(u-\mu)^2}{2}}$. Finally, the symbol $\xrightarrow{a.s.}$ denotes almost sure convergence, $O(\cdot)$ is the big O notation, $\lambda_i(\cdot)$ is the i -th eigenvalue of the con-

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sidered matrix and the symbol $\mathbb{E}_{\mathbf{u}|\mathbf{w}}$ refers to the expectation with respect to $p(\mathbf{u}|\mathbf{w})$.

II. BAYESIAN LINEAR MODEL CORRUPTED BY NOISE OUTLIERS

A. Definition of the random model

Let \mathbf{y} be the $N \times 1$ vector of measurements. The BHLM is defined by

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}, \quad (1)$$

where each element $[\mathbf{A}]_{i,j}$ of the $N \times K$ matrix \mathbf{A} , with $K < N$, is drawn i.i.d. as a single realization of a sub-Gaussian distribution with zero-mean and variance $1/N$ [22,25]. The unknown amplitude vector is given by

$$\mathbf{x} = [x_1, \dots, x_K]^T \sim \mathcal{N}(\mathbf{0}_{K \times 1}, \sigma_x^2 \mathbf{I}_K), \quad (2)$$

where σ_x^2 is the known amplitude variance. In addition, the measurements are contaminated by a noise vector \mathbf{e} which is assumed statistically independent from \mathbf{x} .

B. Hierarchical Normal-Gamma representation

The i -th noise sample is assumed to be circular centered i.i.d. Gaussian according to

$$e_i|\gamma \sim \mathcal{N}\left(0, \frac{\sigma^2}{\gamma}\right), \quad (3)$$

where $\frac{\gamma}{\sigma^2}$ is usually called the noise precision, γ is an unknown hyper-parameter and σ^2 is a fixed scale parameter.

If the hyper-parameter is Gamma distributed according to

$$\gamma \sim \mathcal{G}\left(\frac{\nu}{2}, \frac{\nu}{2}\right), \quad (4)$$

where ν is the number of degrees of freedom, the joint distribution of (e_i, γ) follows a Normal-Gamma distribution [26] such as

$$(e_i, \gamma) \sim \text{NormalGamma}\left(0, \frac{1}{\sigma^2}, \frac{\nu}{2}, \frac{\nu}{2}\right). \quad (5)$$

The marginal distribution of the joint pdf over the hyper-parameter γ leads to a non-standardized Student's t -distribution, given by [11,27]

$$\mathcal{S}(e_i|0, \sigma^2, \nu) = \int_0^\infty \mathcal{N}\left(e_i|0, \frac{\sigma^2}{\gamma}\right) \mathcal{G}\left(\gamma|\frac{\nu}{2}, \frac{\nu}{2}\right) d\gamma, \quad (6)$$

such that $e_i \sim \mathcal{S}(0, \sigma^2, \nu)$.

As $\nu \rightarrow \infty$, the distribution tends to a Gaussian with zero-mean and variance σ^2 , while it becomes more heavy-tailed when ν is small [12,28]. With (3) and (4), and knowing that $\frac{1}{\gamma} \sim \mathcal{IG}(\frac{\nu}{2}, \frac{\nu}{2})$, we notice that the variance, noted σ_e^2 of each noise entry of \mathbf{e} , is given by the following expression

$$\sigma_e^2 = \mathbb{E}_\gamma \mathbb{E}_{e_i|\gamma} \{e_i^2\} = \sigma^2 \mathbb{E}_\gamma \left\{ \frac{1}{\gamma} \right\} = \sigma^2 \frac{\nu}{\nu - 2}, \quad (7)$$

in which $\nu > 2$.

III. BCRB FOR STUDENT'S T-DISTRIBUTION

The vector of unknown parameters, denoted by $\boldsymbol{\theta}$, encompasses the amplitude vector and the noise hyper-parameter, i.e.

$$\boldsymbol{\theta} = [\mathbf{x}^T, \gamma]^T. \quad (8)$$

Given an independence assumption between \mathbf{x} and γ , the joint pdf $p(\mathbf{y}, \boldsymbol{\theta})$ can be decomposed as

$$p(\mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta}) = p(\mathbf{y}|\boldsymbol{\theta})p(\mathbf{x})p(\gamma). \quad (9)$$

Let us note $\hat{\boldsymbol{\theta}}$ an estimator of the unknown vector $\boldsymbol{\theta}$. Then, the mean square error (MSE), directly linked to the error covariance matrix, verifies the following inequality

$$\text{MSE}(\boldsymbol{\theta}) = \text{Tr} \left[\mathbb{E}_{\mathbf{y}, \boldsymbol{\theta}} \left\{ (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T \right\} \right] \geq \text{Tr}[\mathbf{C}], \quad (10)$$

where \mathbf{C} is the $(K+1) \times (K+1)$ BCRB matrix defined as the inverse of the Bayesian information matrix (BIM) \mathbf{J} . We can show that the BIM has a block-diagonal structure due to the independence between parameters. Thus, we write

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{\mathbf{x}, \mathbf{x}} & \mathbf{0}_{K \times 1} \\ \mathbf{0}_{1 \times K} & J_{\gamma, \gamma} \end{bmatrix}. \quad (11)$$

We assume an identifiable BHLM model so that, under weak regularity conditions [19], the BIM is given by

$$\mathbf{J} = \mathbb{E}_{\boldsymbol{\theta}} \left\{ \mathbf{J}_D^{(\boldsymbol{\theta}, \boldsymbol{\theta})} \right\} + \mathbf{J}_P^{(\boldsymbol{\theta}, \boldsymbol{\theta})} + \mathbf{J}_{HP}^{(\boldsymbol{\theta}, \boldsymbol{\theta})}, \quad (12)$$

in which

$$[\mathbf{J}_D^{(\boldsymbol{\theta}, \boldsymbol{\theta})}]_{i,j} = \mathbb{E}_{\mathbf{y}|\boldsymbol{\theta}} \left\{ -\frac{\partial^2 \log p(\mathbf{y}|\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right\}, \quad (13)$$

$$[\mathbf{J}_P^{(\boldsymbol{\theta}, \boldsymbol{\theta})}]_{i,j} = \mathbb{E}_{\mathbf{x}} \left\{ -\frac{\partial^2 \log p(\mathbf{x})}{\partial \theta_i \partial \theta_j} \right\}, \quad (14)$$

$$[\mathbf{J}_{HP}^{(\boldsymbol{\theta}, \boldsymbol{\theta})}]_{i,j} = \mathbb{E}_{\gamma} \left\{ -\frac{\partial^2 \log p(\gamma)}{\partial \theta_i \partial \theta_j} \right\} \quad (15)$$

for $(i, j) \in \{1, \dots, K+1\}^2$, and where $\mathbf{J}_D^{(\boldsymbol{\theta}, \boldsymbol{\theta})}$ is the Fisher information matrix (FIM) on $\boldsymbol{\theta}$, $\mathbf{J}_P^{(\boldsymbol{\theta}, \boldsymbol{\theta})}$ is the prior part of the BIM and $\mathbf{J}_{HP}^{(\boldsymbol{\theta}, \boldsymbol{\theta})}$ is the hyper-prior part.

Correspondingly, we have

$$\mathbf{C} = \mathbf{J}^{-1} = \begin{bmatrix} \mathbf{C}_{\mathbf{x}, \mathbf{x}} & \mathbf{0}_{K \times 1} \\ \mathbf{0}_{1 \times K} & C_{\gamma, \gamma} \end{bmatrix}. \quad (16)$$

Conditionally to $\boldsymbol{\theta}$, the observation vector \mathbf{y} has the following Gaussian distribution

$$\mathbf{y}|\boldsymbol{\theta} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{R}), \quad (17)$$

where $\boldsymbol{\mu} = \mathbf{A}\mathbf{x}$ and $\mathbf{R} = \frac{(\nu-2)\sigma_e^2}{\nu\gamma} \mathbf{I}_N$. In what follows, we directly make use of the Slepian-Bangs formula [29, p. 378]

$$[\mathbf{J}_D^{(\boldsymbol{\theta}, \boldsymbol{\theta})}]_{i,j} = \left(\frac{\partial \boldsymbol{\mu}}{\partial \theta_i} \right)^T \mathbf{R}^{-1} \frac{\partial \boldsymbol{\mu}}{\partial \theta_j} + \frac{1}{2} \text{Tr} \left[\frac{\mathbf{R}}{\partial \theta_i} \mathbf{R}^{-1} \frac{\mathbf{R}}{\partial \theta_j} \mathbf{R}^{-1} \right]. \quad (18)$$

This leads to

$$\mathbf{J}_D^{(\mathbf{x}, \mathbf{x})} = \frac{\nu\gamma}{(\nu-2)\sigma_e^2} \mathbf{A}^T \mathbf{A}. \quad (19)$$

Using the fact that $\mathbf{R}^{-1} = \frac{\gamma}{\sigma^2} \mathbf{I}_N$, we obtain

$$J_D^{(\gamma, \gamma)} = \frac{\sigma^4}{2\gamma^4} \text{Tr}[\mathbf{R}^{-2}] = \frac{N}{2\gamma^2}. \quad (20)$$

According to (2) and considering independent amplitudes, we have

$$-\log p(\mathbf{x}) = \sum_{i=1}^K \left(\frac{1}{2} \log(2\pi\sigma_x^2) + \frac{x_i^2}{2\sigma_x^2} \right). \quad (21)$$

Consequently,

$$\mathbf{J}_P^{(\mathbf{x}, \mathbf{x})} = \frac{1}{\sigma_x^2} \mathbf{I}_K. \quad (22)$$

The BIM \mathbf{J} is therefore composed of the following terms:

$$\mathbf{J}_{\mathbf{x}, \mathbf{x}} = \mathbb{E}_\gamma \left\{ \mathbf{J}_D^{(\mathbf{x}, \mathbf{x})} \right\} + \mathbf{J}_P^{(\mathbf{x}, \mathbf{x})}, \quad (23)$$

$$\mathbf{J}_{\gamma, \gamma} = \mathbb{E}_\gamma \left\{ \mathbf{J}_D^{(\gamma, \gamma)} \right\} + J_{HP}^{(\gamma, \gamma)}. \quad (24)$$

The hyper-prior part of the BIM is given by

$$J_{HP}^{(\gamma, \gamma)} = \mathbb{E}_\gamma \left\{ -\frac{\partial^2 \log p(\gamma)}{\partial \gamma^2} \right\} = \frac{\nu - 2}{2} \mathbb{E}_\gamma \left\{ \frac{1}{\gamma^2} \right\}. \quad (25)$$

The second-order moment of an inverse-Gamma distributed random variable is given by

$$\mathbb{E}_\gamma \left\{ \frac{1}{\gamma^2} \right\} = \frac{\nu^2}{(\nu - 2)(\nu - 4)}, \quad (26)$$

where $\nu > 4$. This finally leads to

$$\mathbf{J}_{\gamma, \gamma} = \frac{N\nu^2}{2(\nu - 2)(\nu - 4)} + \frac{\nu^2}{2(\nu - 4)}. \quad (27)$$

Inverting the BIM, we obtain the BCRB for the amplitude parameters

$$\text{BCRB}(\mathbf{x}) = \frac{\text{Tr}[\mathbf{C}_{\mathbf{x}, \mathbf{x}}]}{K} \quad \text{with } \mathbf{C}_{\mathbf{x}, \mathbf{x}} = \sigma_x^2 (r\mathbf{A}^T \mathbf{A} + \mathbf{I}_K)^{-1}, \quad (28)$$

where $r = \text{SNR} \frac{\nu}{\nu - 2}$ with $\text{SNR} = \frac{\sigma_x^2}{\sigma_e^2}$ (signal-to-noise ratio).

IV. BCRB IN THE ASYMPTOTIC FRAMEWORK

Using (27), the asymptotic BCRB for the hyper-parameter γ is given by

$$C_{\gamma, \gamma} = \frac{1}{J_{\gamma, \gamma}} \xrightarrow{N \rightarrow \infty} C_{\gamma, \gamma}^\infty, \quad (29)$$

where

$$C_{\gamma, \gamma}^\infty = \frac{2(\nu - 4)}{\nu^2}. \quad (30)$$

A. RMT framework

In this section, we consider the context of large random matrices, *i.e.*, for $K, N \rightarrow \infty$ with $\frac{K}{N} \rightarrow \beta \in (0, 1)$. The derived BCRB in this context is the asymptotic normalized BCRB defined by

$$\text{BCRB}(\mathbf{x}) \xrightarrow{a.s.} \text{BCRB}^\infty(\mathbf{x}). \quad (31)$$

Using (28) with [21, p. 11], we obtain

$$\text{BCRB}^\infty(\mathbf{x}) = \sigma_x^2 \left(1 - \frac{f(r, \beta)}{4r\beta} \right) \quad (32)$$

and $f(r, \beta) = \left(\sqrt{r(1 + \sqrt{\beta})^2 + 1} - \sqrt{r(1 - \sqrt{\beta})^2 + 1} \right)^2$.

B. Limit analytical expressions

- For $\beta \ll 1$, *i.e.*, $K \ll N$, after some manipulations and discarding the terms of order superior or equal to $O(\beta^2)$, we obtain

$$f(r, \beta) \approx \frac{4\beta r^2}{r + 1}. \quad (33)$$

Therefore, an asymptotic analytical expression of the BCRB, in the RMT framework, is given by

$$\text{BCRB}^\infty(\mathbf{x}) \approx \frac{\sigma_x^2}{r + 1} = \frac{(\nu - 2)\sigma_x^2}{\nu(1 + \text{SNR}) - 2}. \quad (34)$$

- For small r , also meaning small SNR, according to the Neumann series expansion [30], we have $(r\mathbf{A}^T \mathbf{A} + \mathbf{I}_K)^{-1} \approx \mathbf{I}_K - r\mathbf{A}^T \mathbf{A}$ if the maximal eigenvalue $\lambda_{\max}(r\mathbf{A}^T \mathbf{A}) < 1$. Observe that $r\lambda_{\max}(\mathbf{A}^T \mathbf{A}) \xrightarrow{a.s.} r(1 + \sqrt{\beta})^2$ [20]–[22]. In addition, if SNR is sufficiently small with respect to $(\nu - 2)/(4\nu)$ then

$$\begin{aligned} \text{BCRB}(\mathbf{x}) &\approx \frac{\sigma_x^2}{K} (\text{Tr}[\mathbf{I}_K] - r\text{Tr}[\mathbf{A}^T \mathbf{A}]) \\ &\xrightarrow{a.s.} \sigma_x^2(1 - r) = \frac{\sigma_x^2}{\nu - 2}(\nu - 2 - \nu\text{SNR}). \end{aligned} \quad (35)$$

- For large r , also meaning large SNR, we have

$$\begin{aligned} \text{BCRB}(\mathbf{x}) &\approx \frac{\sigma_x^2}{rK} \left(\text{Tr}[(\mathbf{A}^T \mathbf{A})^{-1}] - \frac{1}{r} \text{Tr}[(\mathbf{A}^T \mathbf{A})^{-2}] \right) \\ &\xrightarrow{a.s.} \frac{\sigma_x^2}{r} \left(\frac{1}{1 - \beta} - \frac{1}{r} \frac{1}{(1 - \beta)^3} \right) \\ &= \frac{(\nu - 2)\sigma_x^2}{\nu\text{SNR}(1 - \beta)} \left(1 - \frac{\nu - 2}{\nu\text{SNR}(1 - \beta)^2} \right), \end{aligned} \quad (36)$$

since [20]–[22]

$$\frac{1}{K} \text{Tr}[(\mathbf{A}^T \mathbf{A})^{-1}] \xrightarrow{a.s.} \frac{1}{1 - \beta}, \quad (37)$$

$$\frac{1}{K} \text{Tr}[(\mathbf{A}^T \mathbf{A})^{-2}] \xrightarrow{a.s.} \frac{1}{(1 - \beta)^3}. \quad (38)$$

C. Comparison between two models with a target common SNR

We consider two different models:

$$(M_0) : \mathbf{y}_0 = \mathbf{A}\mathbf{x} + \mathbf{e}_0 \quad \text{with } e_{i_0} \sim \mathcal{S}(0, \sigma_0^2, \nu_0), \quad (39)$$

$$(M_1) : \mathbf{y}_1 = \mathbf{A}\mathbf{x} + \mathbf{e}_1 \quad \text{with } e_{i_1} \sim \mathcal{S}(0, \sigma_1^2, \nu_1). \quad (40)$$

Model (M_0) is the reference model and model (M_1) is the alternative one. According to (32), the asymptotic normalized BCRB for the k -th model with $k \in \{0, 1\}$ is defined by

$$\text{BCRB}_k^\infty(\mathbf{x}) = \sigma_x^2 \left(1 - \frac{f(r_k, \beta)}{4r_k\beta} \right) \quad (41)$$

where $r_k = \text{SNR}_k \frac{\nu_k}{\nu_k - 2}$ with $\text{SNR}_k = \frac{\sigma_x^2}{\sigma_k^2}$. A fair methodology to compare the bounds $\text{BCRB}_0(\mathbf{x})$ and $\text{BCRB}_1(\mathbf{x})$ is to impose a common target SNR for the models (M_0) and (M_1) , *i.e.*, $\text{SNR}_0 = \text{SNR}_1$. A simple derivation shows that to reach

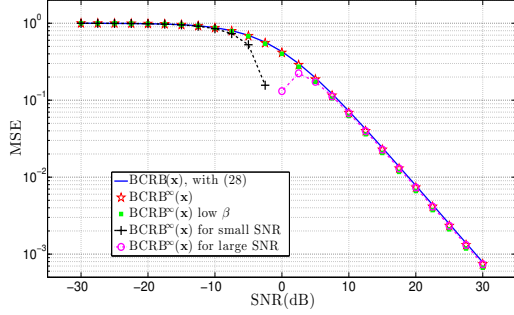


Fig. 1. BCRB(\mathbf{x}) as a function of SNR in dB with specific limit approximations, in the RMT framework.

the target SNR, we must have $r_1 = \frac{\nu_1(\nu_0-2)}{\nu_0(\nu_1-2)}r_0$. Specifically, the corresponding BCRBs are the following ones:

$$\text{BCRB}_0^\infty(\mathbf{x}) = \sigma_x^2 \left(1 - \frac{f(r_0, \beta)}{4r_0\beta} \right), \quad (42)$$

$$\text{BCRB}_1^\infty(\mathbf{x}) = \sigma_x^2 \left(1 - \frac{\nu_0(\nu_1-2)f\left(\frac{\nu_1(\nu_0-2)}{\nu_0(\nu_1-2)}r_0, \beta\right)}{4\nu_1(\nu_0-2)r_0\beta} \right). \quad (43)$$

Recall that the Student's t-distribution is well known to promote noise outliers thanks to its heavy-tails property. Conversely, the Gaussian distribution does not share this behavior. So, an interesting scenario arises when $\nu_1 \rightarrow \infty$. In this case, the Student's t-distribution converges to the Gaussian one [10] and (43) tends to

$$\text{BCRB}_1^\infty(\mathbf{x}) \stackrel{\nu_1 \rightarrow \infty}{=} \sigma_x^2 \left(1 - \frac{\nu_0 f\left(\frac{\nu_0-2}{\nu_0}r_0, \beta\right)}{4(\nu_0-2)r_0\beta} \right). \quad (44)$$

D. Numerical simulations

In the following simulations, we consider $N = 100$ and $K = 10$ so that $\beta \ll 1$. The amplitude variance σ_x^2 is fixed to 1. In Fig. 1, we plot the BCRB of the amplitude vector \mathbf{x} , as defined by equations (28) and (32) (asymptotic expression), (34) (small β), (35) (small SNR) and (36) (large SNR), as a function of the SNR in dB. We choose $\nu = 6$.

We notice that BCRB(\mathbf{x}) coincides precisely with its asymptotic expression in (32). Thus, the RMT framework predicts precisely the behavior of the BCRB of the amplitude as $K, N \rightarrow \infty$ with $\frac{K}{N} \rightarrow \beta$ and allows us to obtain a closed-form expression. Such limit remains correct even for values of N and K that are relatively not quite large. The expression of the BCRB obtained with (34) is a good approximation since here, we have $\beta = 0.1 \ll 1$. Finally, we notice that the curves obtained for low and high SNR approximate very well the BCRB of the amplitude, asymptotically.

In Fig. 2, as exposed in section IV-C, we consider two different models, with a different value for the number of degrees of freedom ν . We notice that a lower performance bound is achieved with $\nu_0 = 6$, especially in the low noise regime, than with $\nu_1 = 100$. Furthermore, the approximation

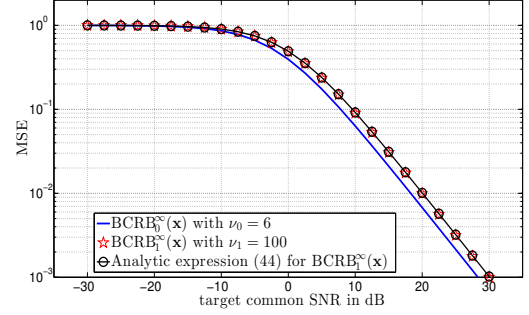


Fig. 2. Asymptotic normalized BCRBs for models (M_0) and (M_1) vs. a common SNR

TABLE I
 $C_{\gamma, \gamma}^\infty$ AS A FUNCTION OF ν .

ν	10	30	50	70	100
$C_{\gamma, \gamma}^\infty$ with (30)	0.1	0.06	0.04	0.03	0.02

in (44) is correct, since ν_1 has a large value. A low value for the number of degrees of freedom is well-adapted for the modelization of sparse (outlier) noise, characterized by a heavy-tailed distribution [31,32]. This large level in heavy-tailedness leads to robustness [1,33,34] while a Gaussian noise model (large degree of freedom) corresponds to a dense noise type. Thus, we can hope to achieve better estimation performances if we consider a model, which promotes sparsity and the presence of outliers in data.

Finally, in Table I, we represent the evolution of $C_{\gamma, \gamma}^\infty$, given in (30), as a function of the number of degrees of freedom ν . We notice that a lower BCRB, for the noise hyper-parameter, is achieved as ν increases, *i.e.*, when the marginal distribution of the noise tends to a Gaussian.

V. CONCLUSION

This work discusses fundamental Bayesian lower bounds for multi-parameter robust estimation. More precisely, we consider a Bayesian linear model corrupted by a sparse noise following a Student's t-distribution. This class of prior can efficiently modelize outliers. Using the hierarchical Normal-Gamma representation of the Student's t-distribution, the Van Trees' Bayesian lower Bound (BCRB) is derived for unknown amplitude parameters and the noise hyper-parameter in an asymptotic context. By asymptotic, it means that the number of measurements and the number of unknown parameters grow to infinity at a finite rate. Consequently, closed-form expressions of the BCRB are obtained using some powerful results from the large Random Matrix Theory (RMT). Finally, a framework is provided to fairly compare two models corrupted by noises with different number of degrees of freedom for a fixed common target SNR. We recall that a small degree of freedom promotes outliers in the sense that the noise prior has heavy-tails. For the amplitude, a lower performance bound is achieved when the number of degrees of freedom is small.

REFERENCES

- [1] A. M. Zoubir, V. Koivunen, Y. Chakhchoukh, and M. Muma, "Robust estimation in signal processing: A tutorial-style treatment of fundamental concepts," *IEEE Signal Processing Magazine*, vol. 29, no. 4, pp. 61–80, 2012.
- [2] K. Mitra, A. Veeraraghavan, and R. Chellappa, "Robust RVM regression using sparse outlier model," in *IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, San Francisco, CA, 2010, pp. 1887–1894.
- [3] —, "Robust regression using sparse learning for high dimensional parameter estimation problems," in *IEEE Int. Conf. Acoust., Speech and Signal Processing (ICASSP)*, Dallas, TX, 2010, pp. 3846–3849.
- [4] P. Zhuang, W. Wang, D. Zeng, and X. Ding, "Robust mixed noise removal with non-parametric Bayesian sparse outlier model," in *16th International Workshop on Multimedia Signal Processing (MMSp)*, Jakarta, Indonesia, 2014, pp. 1–5.
- [5] G. E. Newstadt, A. O. Hero, and J. Simmons, "Robust spectral unmixing for anomaly detection," in *IEEE Workshop on Statistical Signal Processing (SSP)*, Gold Coast, VIC, 2014, pp. 109–112.
- [6] M. Sundin, S. Chatterjee, and M. Jansson, "Combined modeling of sparse and dense noise improves Bayesian RVM," in *22nd European Signal Processing Conference (EUSIPCO)*, Lisbon, Portugal, 2014, pp. 1841–1845.
- [7] —, "Bayesian learning for robust Principal Component Analysis," in *23rd European Signal Processing Conference (EUSIPCO)*, Nice, France, 2015, pp. 2361–2365.
- [8] J. Luttinen, A. Iljin, and J. Karhunen, "Bayesian robust PCA of incomplete data," *Neural Processing Letters*, vol. 36, no. 2, pp. 189–202, 2012.
- [9] D. Peel and G. J. McLachlan, "Robust mixture modelling using the t distribution," *Statistics and computing*, vol. 10, no. 4, pp. 339–348, 2000.
- [10] S. Kotz and S. Nadarajah, *Multivariate t-distributions and their applications*. Cambridge University Press, 2004.
- [11] J. Christmas, "Bayesian spectral analysis with Student-t noise," *IEEE Transactions on Signal Processing*, vol. 62, no. 11, pp. 2871–2878, 2014.
- [12] H. Zhang, Q. M. J. Wu, T. M. Nguyen, and X. Sun, "Synthetic aperture radar image segmentation by modified Student's t-mixture model," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 52, no. 7, pp. 4391–4403, 2014.
- [13] Q. Wei, N. Dobigeon, and J.-Y. Tourneret, "Bayesian fusion of hyperspectral and multispectral images," in *IEEE Int. Conf. on Acoust., Speech and Signal Processing (ICASSP)*, Florence, Italy, 2014, pp. 3176–3180.
- [14] N. L. Pedersen, C. N. Manchón, D. Shutin, and B. H. Fleury, "Application of Bayesian hierarchical prior modeling to sparse channel estimation," in *IEEE International Conference on Communications (ICC)*, Ottawa, ON, 2012, pp. 3487–3492.
- [15] G. Kail, J.-Y. Tourneret, F. Hlawatsch, and N. Dobigeon, "Blind deconvolution of sparse pulse sequences under a minimum distance constraint: A partially collapsed Gibbs sampler method," *IEEE Transactions on Signal Processing*, vol. 60, no. 6, pp. 2727–2743, 2012.
- [16] N. Dobigeon, J.-Y. Tourneret, and J. D. Scargle, "Joint segmentation of multivariate astronomical time series: Bayesian sampling with a hierarchical model," *IEEE Transactions on Signal Processing*, vol. 55, no. 2, pp. 414–423, 2007.
- [17] A. Gelman, "Prior distributions for variance parameters in hierarchical models (comment on article by Browne and Draper)," *Bayesian Analysis*, vol. 1, no. 3, pp. 515–534, 2006.
- [18] J. Dahlin, F. Lindsten, T. B. Schön, and A. Wills, "Hierarchical Bayesian ARX models for robust inference," in *16th IFAC Symposium on System Identification (SYSID)*, Brussels, Belgium, 2012, pp. 131–136.
- [19] H. L. Van Trees and K. L. Bell, *Bayesian bounds for parameter estimation and nonlinear filtering/tracking*. New York: Wiley-IEEE Press, 2007.
- [20] J. W. Silverstein and Z. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," *Journal of Multivariate analysis*, vol. 54, no. 2, pp. 175–192, 1995.
- [21] A. M. Tulino and S. Verdú, *Random matrix theory and wireless communications*. Foundations and Trends in Communications and Information Theory. Now Publishers Inc., 2004, vol. 1, no. 1.
- [22] R. Couillet and M. Debbah, *Random matrix methods for wireless communications*. Cambridge University Press, 2011.
- [23] M. N. El Korso, R. Boyer, P. Larzabal, and B.-H. Fleury, "Estimation performance for the Bayesian hierarchical linear model," *IEEE Signal Processing Letters*, vol. 23, no. 4, pp. 488–492, 2016.
- [24] R. Prasad and C. R. Murthy, "Cramér-Rao-type bounds for sparse Bayesian learning," *IEEE Transactions on Signal Processing*, vol. 61, no. 3, pp. 622–632, 2013.
- [25] V. V. Buldygin and Y. V. Kozachenko, *Metric characterization of random variables and random processes*. American Mathematical Soc., 2000, vol. 188.
- [26] J. M. Bernardo and A. F. M. Smith, *Bayesian theory*. New York: J. Wiley, 1994.
- [27] M. Svensén and C. M. Bishop, "Robust Bayesian mixture modelling," *Neurocomputing*, vol. 64, pp. 235–252, 2005.
- [28] G. Sfikas, C. Nikou, and N. Galatsanos, "Robust image segmentation with mixtures of Student's t-distributions," in *International Conference on Image Processing (ICIP)*, vol. 1, Sant Antonio, TX, 2007, pp. 1–273 – 1–276.
- [29] P. Stoica and R. L. Moses, *Spectral analysis of signals*. Pearson Prentice Hall, Upper Saddle River, NJ, 2005.
- [30] G. B. Arfken and H. J. Weber, *Mathematical methods for physicists, sixth edition*. Academic press, 2005.
- [31] G. Tzagarakis and P. Tsakalides, "Bayesian compressed sensing of a highly impulsive signal in heavy-tailed noise using a multivariate Cauchy prior," in *17th European Signal Processing Conference*, Glasgow, Scotland, 2009, pp. 2293–2297.
- [32] A. Amini, M. Unser, and F. Marvasti, "Compressibility of deterministic and random infinite sequences," *IEEE Transactions on Signal Processing*, vol. 59, no. 11, pp. 5193–5201, 2011.
- [33] K. L. Lange, R. J. A. Little, and J. M. G. Taylor, "Robust statistical modeling using the t distribution," *Journal of the American Statistical Association*, vol. 84, no. 408, pp. 881–896, 1989.
- [34] N. Delannay, C. Archambeau, and M. Verleysen, "Improving the robustness to outliers of mixtures of probabilistic PCAs," *Advances in Knowledge Discovery and Data Mining*, vol. 5012, pp. 527–535, 2008.